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DIFFERENTIAL GEOMETRY IN THE LARGE*

BY D. J. STRUIK

1. *Introduction.* As a typical example of a theorem in this type of differential geometry we take the so-called *four-vertex theorem.* It states that *every closed plane convex curve has at least four different points of extreme curvature.* Thus the ellipse has exactly four such points. Since they are called *vertices,* we use that name for points of extreme curvature in general.

Curvature is a property of differential geometry. Its existence is established by the application of differential calculus to geometry. In the four-vertex theorem a relation is found between the curvature of a general type of curve at points at finite distance. It is obvious that the differential calculus alone cannot lead to such theorems; we need the integral calculus. Therefore, we might call this type of geometry integral geometry. But it has already become customary to call it *differential geometry in the large* (Differentialgeometrie im Grossen).

2. *Proof of the Four-Vertex Theorem.* We shall now give a proof of the four-vertex theorem, to show the nature of such demonstrations. We take a closed plane convex curve having two, and only two, tangents in every direction. We may use this property as a definition, and call such a curve an *oval.* If, however, we define a plane closed convex curve as a continuous closed curve with no more than two points in common with a straight line, we must investigate the relation between both definitions.† Still other ways can be found to define convexity.

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* An address presented at the invitation of the program committee at the meeting of the Society in New York City, April 18, 1930, as part of a symposium on differential geometry.

† See A. Emch, American Journal of Mathematics, vol. 35 (1913), p. 407; S. Nakajima, Tôhoku Mathematical Journal, vol. 29 (1928), p. 227.

Let $\boldsymbol{r}=\boldsymbol{r}(s)$ be the vector equation of the oval, $\boldsymbol{r}$ having the components x and y, while s is the arc length measured from a point on the oval. If $\boldsymbol{t}$ and $\boldsymbol{n}$ are the unit vectors in the tangent and normal directions, uniquely defined by the assumption of a sense of direction on the oval and a sense of rotation in the plane, we have

$$\frac{d\boldsymbol{r}}{ds}=\boldsymbol{t};\quad \frac{d\boldsymbol{t}}{ds}=k\boldsymbol{n};\quad \frac{d\boldsymbol{n}}{ds}=-k\boldsymbol{t},$$

where $k=1/\rho$ is the curvature. Then

$$\int_{s_1}^{s_2}\boldsymbol{r}dk=\boldsymbol{r}k\Big]_{s_1}^{s_2}-\int_{s_1}^{s_2}kd\boldsymbol{r}$$

$$=\boldsymbol{r}k\Big]_{s_1}^{s_2}-\int_{s_1}^{s_2}k\boldsymbol{t}ds=\boldsymbol{r}k+\boldsymbol{n}\Big]_{s_1}^{s_2};$$

therefore, if the integral is taken around the curve

$$\int\boldsymbol{r}dk=0.$$

Now $k(s)$ has certainly one maximum and one minimum, if the oval is not a circle. If we take the origin at one of the points where $dk=0$, we see immediately from the figure that the assumption that there is only one other point for which $dk=0$ leads to an absurdity. The ellipse has four vertices, therefore every oval has at least four vertices.

Among the theorems closely connected with the four-vertex theorem we mention the following:

*On every oval there are at least three pairs of points with parallel tangents and equal curvatures.**

If an oval has n vertices, a circle has at most n points in common with the oval.

In space we can define an *ovaloid* as a closed continuous surface with two and no more than two tangent planes in every direction. A theorem analogous to the four-vertex theorem has for several years been known as the conjecture of Carathéodory.

* W. Blaschke, Archiv der Mathematik und Physik, vol. 26 (1917), p. 65. G. Szegö, ibid., vol. 28 (1920), p. 183. W. Süss, Jahresbericht der Vereinigung, vol. 33 (1924), Part 2, p. 32; Tôhoku Mathematical Journal, vol. 28 (1927), p. 216. T. Hayashi, Palermo Rendiconti, vol. 50 (1926), p. 96.

It is that on every ovaloid there are at least two umbilical points. The exact formulation and demonstration here is much more complicated than in the plane case, but it seems that the proof has recently been given by Cohn-Vossen, for any closed analytic surface that is uniquely representable on a sphere.*

A generalization of the four-vertex theorem for curves in space is also known.†

3. *Affine Properties.* The four-vertex theorems belong to the group of euclidean motions. This suggests the possibility of similar theorems for other groups, as the affine group, the Möbius group of conformal transformations, the group of analysis situs. The conception of a closed convex curve is affine, but instead of osculating circles we have osculating conic sections. The minimum number of points with stationary osculating conic section (the so-called sextactic points) is six.‡ This is similar to the four-vertex theorem. There is also a theorem suggested by the property that a straight line, that is, a curve of zero curvature, has no more than two points in common with an oval. The curve of zero affine curvature is the parabola. The theorem states that if at all points of an oval the osculating conic section is an ellipse (so-called elliptic points), the oval has at most four points in common with a parabola. Another way to state it is that every five points on an elliptically curved oval lie on an ellipse.§ The osculating ellipse of an elliptically curved oval with maximum area encloses the given oval; that with minimum area lies entirely within the oval.¶

4. *Projective Properties.* To differential geometry in the large of the projective group there belong a series of investigations which were started by Kneser‖ in 1888 and which were con-

* W. Blaschke, Mathematische Zeitschrift, vol. 24 (1926), p. 617; see also his *Differentialgeometrie*, vol. 3, p. 289; W. Süss, Tôhoku Mathematical Journal, vol. 27 (1926), p. 306; St. Cohn-Vossen, Proceedings Bologna, 1928; see H. Hamburger, Sitzungsberichte Berliner Akademie, 1922, p. 258; Mathematische Zeitschrift, vol. 19 (1924), p. 50.

† W. Süss, Tôhoku Mathematical Journal, vol. 29 (1928), p. 359.

‡ W. Blaschke, Leipziger Berichte, vol. 69 (1917), p. 321, and *Differentialgeometrie*, vol. 2, p. 43. The theorem is ascribed to G. Herglotz and J. Radon.

§ P. Böhmer, Mathematische Annalen, vol. 60 (1905), p. 256.

¶ S. Ogiwara, Tôhoku Science Reports, vol. 15 (1926), p. 503.

‖ A. Kneser, Mathematische Annalen, vol. 31 (1888), p. 507; vol. 34 (1889), p. 204.

tinued by Juel.* They deal with points of inflexion, osculation planes, and other projective invariants. One of Kneser's simplest results is that a plane arc free from singularities that has no more than two points in common with a straight line, does not send more than two tangents through one point of the plane. Juel remarks that the classification of algebraic curves in curves of second, third, etc., degree allows a generalization; and he studies the properties of plane curves that have no more than three, or no more than four, points in common with a straight line, and analogous curves in space. Such curves, if they are composed of a finite number of convex arcs whose tangents turn continuously, are called *elementary curves*. The maximum number of points of intersection with a line is called their *order*. Among Juel's theorems we find, for example, the following:

The number of points of inflexion of an elementary curve is even or odd, if the order of the curve is even or odd.

An elementary curve of order four consisting of only one branch and without double points and cusps has at least one double tangent.

Through stereographic projection the osculation circles of a plane curve pass into the intersections of the sphere with the osculation planes of the image of the curve. In this way, projective properties may pass into metrical properties. Kneser found the four-vertex theorem in this manner.

5. *The Topological Group.* Several types of differential geometry in the large are known that are invariant under topological transformations. One of the best known is the theorem on the curvatura integra of a closed surface of connectivity p. It states† that the integral of the Gaussian curvature taken over the entire surface depends only on p: $\int R d\sigma = 4\pi(1-p)$. It is one of the applications of Stokes' integral theorem, which, just as Gauss's analogous theorem, may be considered as a property of differential geometry in the large. Extensions of this

* C. Juel, Det danske Videnskabernes Selskabs Skrifter, 1906, p. 297; Jahresbericht der Vereinigung, vol. 16 (1907), p. 196; Mathematische Annalen, vol. 76 (1915), p. 343. At the meeting, the importance of Juel's results was pointed out by S. Lefschetz. See also H. Brunn, Jahresbericht der Vereinigung, vol. 3 (1892–93), p. 84.

† See W. Blaschke, *Differentialgeometrie*, vol. 1, p. 111.

theorem to spaces of constant curvature and of higher dimensions, are known.*

Here belong also the theorems on the behavior in the large of geodesics on surfaces, the Jacobi condition, and Poincaré's theorem that there is at least one closed geodesic on an ovaloid,† it being plausible that there are at least three. On a closed surface of positive curvature a point describing a geodesic must cross any existing closed geodesic an infinite number of times, so that two closed geodesics necessarily intersect.‡ The problem of closed geodesics is of importance in dynamics, where it leads to periodic motions. Birkhoff and Morse have developed methods under which this problem, and that of extremals in variational problems in general, can be studied in more dimensions and on manifolds of different connectivity.

The Jacobi condition for geodesics on a surface, which associates with a given point on a geodesic another point, called the *conjugate* point, gives rise to the theorem that the locus of the points conjugate to a point on an ovaloid has at least four cusps. On an ellipsoid there are exactly four cusps. It is not yet satisfactorily proved.§

In Riemannian manifolds of more than two dimensions hardly anything except local differential geometry has been studied. An exception might be made for a generalization of the theorem that only on surfaces of constant curvature curves of constant geodesic distance to a point (so-called distance circles) and curves of constant geodesic curvature (so-called curvature circles) are identical. Only in manifolds of constant curvature exist closed curvature spheres that can be contracted to a point without losing their property.‖

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* H. Hopf, Göttinger Nachrichten, 1925, p. 131; Mathematische Annalen, vol. 95 (1925), p. 340; these papers discuss the problem from the point of view of analysis situs. The subject was discussed from the point of view of differential geometry by D. J. Struik in a paper read to the Harvard colloquium in the Spring of 1930.

† H. Poincaré, Transactions of this Society, vol. 6 (1905), p. 237; see also W. Blaschke, *Differentialgeometrie*, vol. 1, p. 142. See A. Speiser, Vierteljahrsschrift Naturforschenden Gesellschaft Zürich, vol. 66 (1921), p. 28.

‡ Hadamard, Journal de Mathématiques, vol. 3 (1897), p. 331. See also M. Kerner, Mathematische Annalen, vol. 101 (1929), p. 633.

§ Blaschke, *Differentialgeometrie*, vol. 1, p. 160, from Carathéodory.

‖ B. Baule, Mathematische Annalen, vol. 83 (1921), p. 286; vol. 84, p. 202.

6. *Circles and Spheres.* The properties of circles and spheres attracted the attention of the geometers perhaps since the time when the Neanderthal man looked up and saw the sky revolving above his head. Many attempts have been made to characterize these figures, either for purely scientific reasons, to find interesting properties, or in an attempt to express their perfection. We give some of those properties here.

(a) *Of all closed plane rectifiable curves of given circumference the circle has maximum area.*

This property has been known from antiquity, and was perhaps suggested by the experience of generals, who found that the number of troops necessary to surround a town was no measure for its extent. The proposition is known as the isoperimetric problem, and has been the starting point for the finding of many inequalities on closed curves.*

(b) *The circle can be turned around in a square and always remain tangent to the four sides.* Curves that have this property are called *curves of constant breadth.*

(c) *A needle of length less than the diameter of the circle can turn through an angle of* 360° *without hitting the circumference.* The problem to find such curves of minimum area is called Kakeya's problem.†

(d) *There exists a point inside the circle for which all chords passing through it are equal. The same point divides all such chords in two equal parts.* This is called the center property. The first property exists for many more curves, for instance the limaçon. We call such a point a distance point. In the same way there exists an area point for closed surfaces. In this case, however, the sphere is uniquely determined by the property that there exists a point such that all plane sections passing through it are equal. The center property, however, is affine, and characterizes ellipses and ellipsoids.‡

(e) *All geodesics on a sphere are closed.* Some other surfaces of this kind are known, both surfaces of revolution, and also others.§

* See W. Blaschke, *Kreis und Kugel*, 1916; and T. Bonnesen, *Les Problèmes des Isopérimètres et des Isépiphanes*, Paris, Gauthier-Villars, 1929. Both books contain bibliographies.

† S. Kakeya, Tôhoku Science Reports, vol. 6 (1917–18), p. 71.

‡ T. Kubota, Tôhoku Science Reports, vol. 3 (1913–14), p. 235.

§ O. Zoll, Mathematische Annalen, vol. 57 (1903), pp. 108–133; G. Darboux, *Leçons sur la Théorie des Surfaces*, vol. 3, p. 4.

(f) *All geodesics on a sphere are plane.*

(g) *The sphere is the only surface of constant non-zero Gaussian curvature which is without singular lines.* This might be expressed by saying that a sphere cannot be deformed without singularities being formed.* For ovaloids an analogous property can be proved, that is, it can be shown that two continuously curved ovaloids, built up by analytical parts, are congruent or symmetrical, if they have the same line element. If, however, a small part of the surface of the ovaloid is taken away, of diameter less than any preassigned number, infinitesimal deformations are possible. An analytic surface of constant negative curvature without singularities is impossible.†

An ovaloid is therefore uniquely determined (but for translations) if the Gaussian curvature is given as a function of the spherical image.‡

(h) *All distance circles are curvature circles.* See §6.

(i) *All geodesics starting from a point pass through another point.* An unfinished attempt to characterize the sphere by this property exists.§

(j) *All circumscribed cylinders are circular cylinders.* This property is characteristic for a sphere.¶ Ovaloids of constant breadth show a generalization.

(k) *The sphere has constant mean curvature.* Among all closed surfaces of unique spherical image, the sphere is characterized by this property.‖

7. *Intensive Study of the Preceding Properties.* We shall study more closely some of these properties. We define the distance of two parallel tangents to an oval as the breadth of this oval at the corresponding points. Then we have for curves

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* H. Liebmann, Münchener Berichte, 1919, p. 267. H. Weyl, Vierteljahrsschrift der Naturforschenden Gesellschaft zu Zürich, p. 40. S. Cohn-Vossen, Göttinger Nachrichten, 1927, p. 125.

† D. Hilbert, Transactions of this Society, vol. 2 (1900), pp. 87–99. For discussion and precision, see L. Bieberbach, Acta Mathematica, vol. 48, p. 319.

‡ H. Minkowski, Gesammelte Abhandlungen, vol. 2 (1911), p. 125.

§ E. B. Christoffel, Werke, vol. 1, p. 162. See S. Nakajima, Tôhoku Mathematical Journal, vol. 28 (1927), p. 266; C. Carathéodory, Abhandlungen aus dem Mathematischen Seminar Hamburg, vol. 4 (1926), p. 297.

¶ W. Blaschke, *Differentialgeometrie*, vol. 1, p. 155.

‖ M. Fujiwara, Tôhoku Science Reports, vol. 3 (1913–14), p. 199.

of constant breadth d the following properties, several of which are due to Euler, who called these curves *orbiform curves.**

(a) *The circumference C of such a curve $=\pi d$.*†

(b) *The line connecting two corresponding points is perpendicular to the tangent and has therefore length d.*

(c) *The sum of the curvatures at corresponding points is constant.*

(d) *The evolute is a curve of zero breadth, that is, a curve of which one and only one tangent has a given direction.* An example is Steiner's hypocycloid.‡ And vice-versa, the involute of a curve of zero breadth is a curve of constant breadth.

There exists a curious relation between the curves of constant breadth and Buffon's needle problem in probability theory. This was, in fact, the reason that Barbier became interested. A needle of length l is thrown upon a table where parallel lines are drawn at distance $a>l$. Then the probability that the needle will hit a line is $2l/(\pi a)$. If we bend the needle into two parts making an arbitrary angle with each other the probability will not change. Reasoning in this way we can see that any needle bent into a closed convex curve of length L will have the chance $L/(\pi a)$ to hit a line as long as the curve can hit at most one line. If, however, the curve hits only one line in one position, and two lines in another, we can obtain a curve of constant breadth with the same probability of hitting. For this we roll the curve along one of the lines and cut off the part of the curve outside of the envelope formed by the different positions of the other parallel line on the plane of the curve.§

For surfaces of constant breadth, defined as ovaloids with their sets of parallel tangent planes at constant distance, we know, besides,¶ that they are surfaces of constant perimeter,

* Euler, Acta Academiae Petropolitanae pro 1778 (1781). See G. Loria, *Spezielle ebene Kurven*, vol. 1, 1910, p. 376, or Jordan-Fiedler, loc. cit.

† E. Barbier, Journal de Mathématiques, (2), vol. 5 (1860), p. 273. Barbier mentions Puiseux as interested in these curves. Euler mentions properties (b) and (d).

‡ See also W. Blaschke, Mathematische Annalen, vol. 76 (1915), p. 504.

§ This reference to the needle problem was given, in the discussion, by O. Ore, who suggested, following Barbier, generalizations where the curve can roll not between two parallel lines but between more general curves.

¶ H. Minkowski, Gesammelte Abhandlungen, vol. 2, p. 277. The inverse property is also true; see W. Blaschke and G. Hessenberg, Jahresbericht der Vereinigung, vol. 26 (1918), p. 215.

which means that the tangent cylinders have cross sections of constant area perpendicular to the generating lines. Such surfaces do not, as a rule, have an area $=\pi d^2$, as the analogy with the plane might suggest. This was discovered by Barbier, who showed, besides, how we can obtain an infinite number of such surfaces.

A generalization of curves of constant breadth can be obtained by studying curves that can turn around in a given regular convex polygon, always remaining tangent to the sides. All such curves inscribed in a polygon of n sides have the same circumference, and at least $2(n-1)$ vertices.* For space curves of constant breadth see §9.

Curves possessing a distance point were investigated in a study on the geometrical form of leaves.† They can be considered as conchoids with respect to themselves. For a similar reason curves of constant breadth are their own parallel curves.

Among the chords of a closed curve in a certain direction, there are one or more of maximum length if the curve is an oval. At the end points of such extremal chords the tangents are parallel. Comparing all extremals in different directions, we have one or more of extremal length. For such a chord the tangents at the end are perpendicular.‡

Kakeya's problem has different answers for different cases. An oval satisfying its condition is the equilateral triangle.§ If we ask for continuous curves only, a solution is Steiner's hypocycloid. But it can be shown that there are point sets of arbitrarily small Jordan measure, in which a line-segment of given length can turn through 360°. Here there is, of course, no longer continuity.¶

8. *Inequalities.* A great number of properties of ovals are inequalities. The starting point of the investigation was the *isoperimetrical* inequality between the circumference C and

* M. Fujiwara, Tôhoku Science Reports, vol. 4 (1915), p. 44.

† B. Habenicht, *Die analytische Form der Blätter* (Quedlinburg, 1905); *Beiträge zur Mathematischen Begründung einer Morphologie der Blätter* (Berlin, 1905); see G. Loria, loc. cit., p. 369.

‡ Hayashi, Tôhoku Mathematical Journal, vol. 22 (1923), p. 387.

§ J. Pál, Mathematische Annalen, vol. 83 (1921), p. 311; compare W. B. Ford, this Bulletin, vol. 28 (1922), p. 45.

¶ A. S. Besicovitch, Journal of the Mathematico-Physical Society of Perm, vol. 2 (1920); Mathematische Zeitschrift, vol. 27 (1928), p. 312.

the area A of a closed curve: $4\pi A \leqq C^2$, where the sign $=$ only holds for the circle. The analogous property for solid bodies is $36\pi V^2 \leqq A^3$, where V is the volume, A the area, and the sign $=$ only holds for the sphere.*

Another simple inequality for ovals is $\pi D \geqq C$, $\pi\Delta \leqq C$. Here D is the maximum distance between two parallel tangents, sometimes called the diameter. The minimum distance Δ is sometimes called the thickness. The sign $=$ holds for curves of constant breadth.†

Another inequality, valid if the oval has a curvature defined at every point, is

$$2\pi R \geqq C, \quad 2\pi r \leqq C,$$

$$\pi R^2 \geqq A, \quad \pi r^2 \leqq A,$$

where R and r are the maximum and minimum radii of curvature,‡ and

$$\Delta \geqq 2r, \quad D \leqq 2R, \quad 4D^2 \geqq C^2 - 4\pi A\,.\,\S$$

There are other inequalities, sharper than those just mentioned.¶

In affine geometry there are inequalities expressing the isoperimetrical property of ellipse and ellipsoid. The principal ones are for ovals:

$$8\pi A \geqq C_1^2,$$

where C_1 is the affine length of the oval

$$C_1 = \int (\dot{\mathbf{r}} \times \ddot{\mathbf{r}})^{1/3} dt;$$

and for ovaloids:

$$12\pi V \geqq A_1^2,$$

where A_1 is the affine area of the ovaloid

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* T. Bonnesen, Mathematische Annalen, vol. 95 (1926), p. 267.

† A. Rosenthal and O. Szasz, Jahresbericht der Vereinigung, vol. 25 (1910). W. Blaschke, Leipziger Berichte, vol. 67 (1915), pp. 290–298.

‡ A. Hurwitz, Annales de l'École Normale, (3), vol. 19 (1902), pp. 357–408. W. Blaschke, *Kreis und Kugel*, 1916.

§ T. Kubota, Tôhoku Science Reports, vol. 13 (1924–25), p. 14.

¶ T. Kubota, Tôhoku Science Reports, vol. 12 (1923–24), p. 45; S. Fukasawa, Tôhoku Mathematical Journal, vol. 26 (1920), p. 27.

$$A_1 = \iint (|\, EG - F^2 \,|)dudv,$$

where $ds^2 = E\,du^2 + 2\,F\,du\,dv + G\,dv^2$.*

The derivation of these inequalities can be closely connected with the conception of mixed area and mixed volume. Take, for instance, two ovaloids, O_1, O_2, given by the vectors, $\boldsymbol{r}_1$, $\boldsymbol{r}_2$. Then the vector $\boldsymbol{r} = \lambda \boldsymbol{r}_1 + \mu \boldsymbol{r}_2$, $\lambda \geqq 0$, $\mu \geqq 0$, determines a third ovaloid O_{12}. Its volume $V(\lambda, \mu)$ can be expressed as a cubic form in λ, μ,

$$V(\lambda, \mu) = V_{111}\lambda^3 + 3V_{112}\lambda^2\mu + 3V_{122}\lambda\mu^2 + V_{222}\mu^3.$$

Here V_{111}, V_{222} are the volumes of O_1,, O_2, and the two quantities V_{112}, V_{122} are called the *mixed volumes* of O_1, O_2. Then we have

$$V_{112}^2 \geqq V_{111}V_{122},$$

$$V_{122}^2 \geqq V_{112}V_{222};$$

and hence

$$V_{112}^3 \geqq V_{111}^2V_{222},$$

where the sign $=$ only holds if O_1, O_2 are similar and in similar position. If O_2 is the unit sphere, the last equality gives the isoperimetrical property of the sphere.†

9. *Space Curves.* Closed space curves have, thus far, not had the attention that we might expect. Still, some interesting properties have been detected, of which we shall mention a few.

A space curve of constant breadth can be obtained by taking a closed curve whose normal plane at a point P has only one more point Q in common with the curve, and for which PQ is constant. For such curves PQ is also normal at Q; the chords PQ form a one-sided surface. Such a curve lies on a surface of constant breadth.‡

It is also possible to define curves of constant breadth on the sphere,§ or in conformal geometry sets of ∞^1 circles in the plane having constant conformal breadth.¶

* See Blaschke, *Differentialgeometrie*, vol. 2; also R. Zindler, Wiener Berichte, vol. 130 (1929), p. 289.

† See H. Minkowski, *Gesammelte Abhundlungen*, vol. 2, p. 250.

‡ M. Fujiwara, Tôhoku Mathematical Journal, vol. 5 (1914), p. 731; vol. 8 (1915), p. 1; W. Blaschke, Leipziger Berichte, vol. 66 (1914), p. 171.

§ W. Blaschke, Leipziger Berichte, vol. 67 (1917), p. 290.

¶ T. Takasu, Tôhoku Science Reports, vol. 17 (1928), pp. 273, 345.

The spherical image of a closed space curve on the unit sphere is a closed spherical curve. It is intersected by every great circle.* If it has at most one double point, the torsion must change its sign, if it does not vanish identically.†

The total curvature $\int k ds$ of a closed space curve is $\geqq 2\pi$, where the sign $=$ holds only for plane convex curves. The quantity k is the (first) curvature of the curve.

If $k =$ constant, we call the curve a skew circle. If we consider skew circles of $k=1$ that are not closed, the angle α between an arc of length $l \leqq \pi$ and its chord is $\leqq l/2$, the sign $=$ only holding for an arc of the unit circle.‡

10. *Other Theorems on Ovals.* Some curious theorems on ovals deal with the so-called *curvature centroid* (Krümmungsschwerpunkt) of plane curves. It is the point in the plane defined by the radius vector $\int \mathbf{r} d\phi$. As integrals along the curve

$$\int_0 \mathbf{r} d\rho = \int_0 \frac{\mathbf{r} ds}{\rho}$$

it can be defined as the center of gravity of the curve if it were loaded with a mass distribution proportional to the curvature. Then we have the following two theorems.

The point with respect to which the pedal curve of an oval is a minimum is the curvature centroid.§

If instead of looking for the center of gravity of the loaded curve we ask for the moments of inertia, we find a tensor $\int \mathbf{r}\mathbf{r} d\phi$. The two principal axes, determined by this tensor, may be called *curvature-axes.*¶ Some properties of ovals with respect to such axes have been found.

Some other curious theorems on ovals are those dealing with inscribed or circumscribed figures. In an oval at least one square can be inscribed. At least two similar rectangles can

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* K. Löwner, in Polya-Szegö, *Aufgaben und Lehrsätze*, vol. 2, 1925, pp. 165, 391.

† W. Fenschel, Mathematische Annalen, vol. 101 (1929), p. 239.

‡ H. A. Schwarz, in Blaschke, *Differentialgeometrie*, vol. 1, p. 47. Here occur more theorems on these circles.

§ J. Steiner, Gesammelte Werke, vol. 2, p. 99.

¶ B. Su, Tôhoku Science Reports, vol. 17 (1928), p. 35, calls *curvature axis* the axis with respect to which the moment of inertia is a minimum.

be inscribed.* The same holds for circumscribed figures.† About an ovaloid a single infinity of cubes can be circumscribed, the points of contact forming continuous curves. In the case of an ellipsoid these cubes are congruent.‡ A certain similarity to these theorems is shown by the following theorem, which holds for any closed rectifiable curve.

There exists always at least one set of four points on the circumference of a closed curve that lie on a circle and divide the circumference in four equal parts.

There are many generalizations of this theorem. For a space curve, for instance, we can prove that a similar property holds for four points in a plane.§ Finally, if two congruent closed plane curves without multiple points coincide with each other, when they have three points in common, they must be circles.¶

11. *History of the Subject.* The history of this subject leads from Zenodor via some occasional work of Euler and Steiner to the end of the nineteenth century. The first important work on convex figures was by Brunn.‖ Then came Minkowski's monumental work.** An interesting little book on plane convex curves was written by Jordan and Fiedler.†† Minkowski's work has been continued by Blaschke, who showed the beauty of differential geometry in the large in many papers and in three books.‡‡ Under his influence not only many German authors wrote on such subjects, but also the present school of Japanese geometers.

The importance of differential geometry in the large was first, so to speak, officially recognized in this country in an

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* A. Emch, American Journal of Mathematics, vol. 36 (1913), p. 407; this Bulletin, vol. 20 (1913), p. 27.

† S. Kakeya, Tôhoku Mathematical Journal, vol. 9 (1916), p. 163.

‡ T. Hayashi, Tôhoku Science Reports, vol. 3 (1913–14), p. 15.

§ Communication to the author by A. Kawaguchi.

¶ T. Kajima, Tôhoku Mathematical Journal, vol. 21 (1922), p. 15; T. Kubota, ibid., p. 21.

‖ Brunn, Dissertation, München, 1887; Münchener Berichte, 1894, p. 102.

** H. Minkowski, Gesammelte Abhandlungen, vol. 2, p. 131.

†† Jordan-Fiedler, *Contributions à l'Étude des Courbes Convexes Fermées*, Paris, Hermann, 1912.

‡‡ W. Blaschke, *Kreis und Kugel*, 1916; *Differentialgeometrie*, vol. 1, and vol. 2. These books contain many references to the literature.

address by Kasner to this Society.* Later came two papers by Emch. A report on the subject has been published by Ball.† The most important contributions to this part of geometry have resulted, in this country, from the work of Birkhoff and Morse. Their starting point, however, is not the same as ours; it is the application of the calculus of variations to dynamics.

12. *Conclusion.* The results so far obtained in differential geometry in the large are not very systematic. The demonstrations often carry an ad hoc character. But they have a striking simplicity which is both charming and stimulating. Their way of discovery, their demonstrations, their general character belong to geometry in its most genuine form. It is certainly not true that the rococo time alone has had the privilege of elegance in its mathematical discoveries. It is not true either that the end of the romantic age of the Holy Alliance has witnessed the end of geometry in its best sense. The work of Minkowski, Blaschke and others is not the feeble work of epigones. If we consider also the splendid development of local differential geometry since the discovery of pseudospherical displacement, it certainly seems as though we were not a hundred years from a heroic age of geometry,but rather just in the midst of another one.

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* E. Kasner, this Bulletin, vol. 11 (1904–05), p. 283.

† N. H. Ball, American Mathematical Monthly, vol. 27 (1930), p. 348.

NOTE ON THE NUMBER OF LINEARLY INDEPENDENT DIRICHLET SERIES THAT SATISFY CERTAIN FUNCTIONAL EQUATIONS*

BY J. I. HUTCHINSON

In a paper published in the Transactions of this Society,† I have determined for each positive integer a the conditions on the coefficients of certain linear combinations of the Dirichlet series

$$\sum_{\nu=0}^{\infty} (a\nu+b)^{-s}, \qquad (b=1, \cdots, a-1),$$

in order that the function $f(s)$ thus determined shall satisfy one or another of the four functional equations

$$\text{(I) and (II):} \quad f(s) = \pm (a)^{1/2} \left(\frac{2\pi}{a}\right)^{s} \sin \frac{\pi s}{2} \frac{\Gamma(1-s)}{\pi} f(1-s),$$

$$\text{(III) and (IV):} \quad f(s) = \pm (a)^{1/2} \left(\frac{2\pi}{a}\right)^{s} \cos \frac{\pi s}{2} \frac{\Gamma(1-s)}{\pi} f(1-s).$$

Three cases had to be considered; $1: a \equiv 2 \pmod 4$, which was completely solved; $2: a=4q$; $3: a=2m+1$. These two cases were carried so far as to determine the roots of the characteristic equations $D(k)=0$ and the probable multiplicities of these roots. As the multiplicity of each root determines the number of linearly independent functions $f(s)$ that satisfy a given one of the equations (I), $\cdots$, (IV), it is evidently important to give a precise solution of the problem.

This seems all the more desirable, since the problem was first solved for the case a a prime number, by E. Cahen in 1894,‡ and needs only to be completed in the one respect indicated to have a complete and rigorous solution for every positive integer value of a.

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* Presented to the Society, September 9, 1930.

† *Properties of functions represented by the Dirichlet series* $\sum(a\nu+b)^{-s}$, *or by linear combinations of such series*, vol. 31 (1929), pp. 322–344. I shall refer to this paper briefly by *T*.

‡ Annales de l'École Normale, (3), vol. 11; in particular, pp. 137–154.

The solution is very simple and consists in observing that the sum of the roots of $D(k)=0$ [T, pp. 333–339] is, in every case, $\sum c_\nu^2$ for the (A) equations [T, p. 332] and $\sum s_\nu^2$ for the (B) equations [T, p. 336],

$$c_\mu = \cos \frac{2\pi\mu}{a}, \qquad s_\mu = \sin \frac{(2\pi\mu}{a} .$$

These may be evaluated by Gauss' sums.

To illustrate: In case 2, $a=4q$, we have*

$$\sum_{\nu=1}^{2q-1} c_\nu^2 = \frac{\sqrt{a}}{2} - 1.$$

The roots of $D(k)$ are [T, p. 338] 0, -1, both simple, and $\sqrt{a}/2$, $-\sqrt{a}/2$ of multiplicities which we will denote by p and n respectively. Then,

$$\text{sum of roots of } D(k) = 0-1+\tfrac{1}{2}(a)^{1/2}(p-n)$$

$$= \frac{\sqrt{a}}{2} - 1 .$$

Moreover, since the degree of $D(k)$ is $2q-1$, we have

$$p + n + 2 = 2q - 1.$$

The solution of these two equations gives

$$p = q - 1, \quad n = q - 2,$$

which agree with the conjectured values [T, p. 328].

All the other cases may be treated in a similar manner and verify the results previously indicated without proof.

CORNELL UNIVERSITY

* See, for example, E. Landau, *Vorlesungen über Zahlentheorie*, vol. I, p. 153.

MODULAR INTERPOLATION*

BY E. T. BELL

1. *Interpolation* mod m. Let m be a constant integer >1, and let $r_1, \cdots, r_s$ denote s distinct constant integers such that

$$0 \leqq r_i < m, \quad (i = 1, \cdots, s, s \leqq m).$$

Let $k_1, \cdots, k_s$ represent any s constants. It is required to construct a function $f(x)$ which is defined when $0 < x \equiv r_i \bmod m$, $(i = 1, \cdots, s)$, and which is such that

$$f(x) = k_i, \text{ when } x \equiv r_i \bmod m, \quad (i = 1, \cdots, s). \tag{1}$$

This $f(x)$ will be called a modular interpolation function for $(r_1, \cdots, r_s) \bmod m$. Such functions occur frequently in the theory of numbers, and are usually constructed "by inspection." A straightforward method of constructing $f(x)$ in (1) is required, as "inspection" is impracticable in only moderately complicated instances.

There obviously is no loss of generality in assuming $x > 0$ as above. For, if x were < 0, a congruence $x \equiv r_i \bmod m$ could be replaced by $-x \equiv m - r_i \bmod m$, which is the previous case.

The general formula for $f(x)$ in §3 in its unreduced form is as simple as Lagrange's for ordinary interpolation. The final reduced polynomial forms of modular interpolation formulas have (in general) simpler coefficients than their correspondents in ordinary interpolation. This is due to reduction by the proper moduli, which has no analog in the ordinary case.

2. *Notation.* The notations of §1 for m, s, r_i, k_i and the following are fixed throughout the paper. We write $\phi(m) = \tau$, the number of integers $\leqq m$ and prime to m; $\mu(1) = 1$, $\mu(m) = 0$ if m is divisible by a square > 1, $\mu(m) = 1$ or -1 in the contrary case according as m is the product of an even or an odd number of primes (Möbius' function). The totitives (positive integers $\leqq m$ and prime to m) of m are denoted by $t_1 = 1, t_2, \cdots, t_\tau$. The polynomial in y of degree τ whose roots are the τ primitive mth roots of unity is denoted by $P(y)$,

* Presented to the Society, November 29, 1930.

$$(2) \qquad P(y) = \prod(y^d - 1)^{\mu(\delta)} = y^\tau + a_1 y^{\tau-1} + \cdots + a_\tau,$$

the product referring to all pairs (d, δ) of positive divisors of m such that $m = d\delta$. The coefficients $a_1, \cdots, a_\tau$ are rational integers.

If ρ is any root of $P(y) = 0$, all the roots are ρ^{ν_j} $(j = 1, \cdots, \tau)$, and if u, v are integers, $\rho^u = \rho^v$ when and only when $u \equiv v \bmod m$.

If in any formula containing ρ it is desired to express everything in terms of m, it is convenient to replace ρ by the particular root $\exp(2\pi i/m)$.

Any rational function $A(\rho)$ of ρ is uniquely reducible to a polynomial in ρ of degree $\tau - 1$ with rational coefficients. If this polynomial is $R(\rho)$, we shall write

$$(3) \qquad A(\rho) \equiv R(\rho) \bmod P(\rho).$$

3. *Unreduced $f(x)$.* Write as definitions of $Q(y)$, $Q_i(y)$,

$$Q(y) = \prod_{i=1}^{s}(y - \rho^{r_i}), \qquad Q_i(y) = Q(y)/(y - \rho^{r_i}).$$

Denote by $Q'(z)$ the result of putting $y = z$ in the derivative $Q'(y)$ of $Q(y)$ with respect to y. Then $f(x)$ in (1) is

$$(4) \qquad f(x) = \sum_{i=1}^{s} \frac{k_i}{\rho^x - \rho^{r_i}} \cdot \frac{Q(\rho^x)}{Q_i(\rho^{r_i})}.$$

For $f(r_i) = k_i$, obviously, and $f(x)$ has the required periodicity mod m. We may write in a form more convenient for reduction to a polynomial in ρ^x,

$$(5) \qquad f(x) = Q(y) \sum_{i=1}^{s} \frac{k_i}{(y - \rho^{r_i})Q'(\rho^{r_i})}, \quad (y = \rho^x).$$

No $Q'(\rho^{r_i}) = 0$, since $Q(y)$ has no repeated root.

4. *Reduced $f(x)$.* At any stage of the following any exponent of ρ may be replaced by its least positive residue mod m, since $\rho^m = 1$. If m is even, $\rho^{m/2} = -1$, and further immediate reduction is possible. The coefficients of the several powers of y will in general be polynomials in ρ; these are reduced as in (3). Write

$$D(\rho) = \prod_{i=1}^{s} Q'(\rho^{r_i}), \qquad D(\rho)/Q'(\rho^{r_i}) = D_i(\rho).$$

Then (5) becomes

$$D(\rho)f(x) = \sum_{i=1}^{s} k_i D_i(\rho) Q_i(y). \tag{6}$$

In this the $D(\rho)$, $D_i(\rho)$ are polynomials in ρ with rational integral coefficients, and the $Q_i(y)$ are polynomials of degree $s-1$ in y whose coefficients are polynomials of the same kind as the $D(\rho)$, $D_i(\rho)$. We may assume that all polynomials in ρ in (6) have been reduced as indicated modulis $P(\rho)$, ρ^m-1 (or $\rho^{m/2}+1$ if m is even), so that all are now of degree $\leqq \tau-1$ in ρ. If $m=2$ the rest of the reduction is obvious. If $m>2$, then $\tau>1$. Multiply throughout by $D(\rho^{t_2}) \cdots D(\rho^{t_\tau})$. The coefficient of $f(x)$ is then a symmetric function of the roots of $P(y)=0$, and hence is a rational integer, say r. The new right-hand member is now multiplied out, reductions by the two moduli being performed as convenient. The final result is

$$rf(x) = \sum_{i=0}^{s-1} R_i(\rho) y^i, \quad (y = \rho^z), \tag{7}$$

where the $R_i(\rho)$ are polynomials in ρ of degree $\leqq \tau-1$ whose coefficients are linear functions of $k_1, \cdots, k_s$ with rational integral coefficients, and r is a rational integer.

Equality being stronger than congruence, either of (5), (7) is a solution of

$$f(x) \equiv k_i \bmod m_i \text{ when } x \equiv r_i \bmod m, \quad (i = 1, \cdots, s), \tag{8}$$

the m_i, k_i being now given constant integers, the m_i different from zero.

5. *Example.* Take $m=8$, $s=4$, $(r_1, r_2, r_3, r_4)=(1, 3, 5, 7)$, $(k_1, k_2, k_3, k_4)=(1, -1, -1, 1)$. The $f(x)$ then defined by (1) is the quadratic character $(2\,|x)$, x odd. Here (2) is $P(y)=y^4+1$, and $\rho^2=1$, $\rho^4=-1$. In §3 we have $Q(y)=P(y)$; hence we have also $Q'(y)=4y^3$, and reduction of (5) gives (7) in the form $2f(x)=\rho(\rho^2-1)\rho^x(\rho^{2x}-1)$. In trigonometric form this gives

$$f(x) = -2i^{x+1} \sin\frac{\pi}{4} \sin\frac{\pi x}{4},$$

where $i=(-1)^{1/2}$.

6. *A Special Reduction.* When $s=\tau$ (see §2) and $(r_1, \cdots, r_s) = (t_1, \cdots, t_\tau)$, the polynomials $P(y)$ in (2) and $Q(y)$ in §3 are

identical. In this case, which is of frequent occurrence, particularly in connection with power residues, a special reduction is sometimes useful.

Let $x \geqq 0$ be an integer. Then

$$\rho^x = f_0(x) + \rho f_1(x) + \cdots + \rho^{\tau-1} f_{\tau-1}(x), \tag{9}$$

where the $f_i(x)$ are uniquely determined rational integers. If x were <0, we should write $\rho^{-x} = \rho^{hm-x}$, h = the least positive integer such that $hm - x \geqq 0$, thus reducing this case to the former.

The τ sequences

$$f_i(x), \qquad (x = 0, 1, \cdots; i = 0, \cdots, \tau - 1), \tag{10}$$

have the characteristic equation $P(y) = 0$,

$$f_i(x + \tau) + a_1 f_i(x + \tau - 1) + \cdots + a_\tau f_i(x) = 0, \tag{11}$$

with the initial values

$$f_i(j) = \delta_{ij}, \qquad (i, j = 0, 1, \cdots, \tau - 1), \tag{12}$$

where $\delta_{ij} = 1$ if $i = j$, $\delta_{ij} = 0$ if $i \neq j$; whence the $f_i(x)$ can be calculated by recurrence. This is often shorter than division by $P(\rho)$.

The general solution of (11), for x restricted to integral values (not necessarily positive), is

$$f(z) = c_1 \rho^{t_1 z} + \cdots + c_\tau \rho^{t_\tau z}, \qquad (z = 0, \pm 1, \pm 2, \cdots).$$

Replacing z by $z + m$, we see that

$$f(z + m) = f(z), \tag{13}$$

since $\rho^m = 1$. Hence in particular the $f_i(x)$ have the period m. If m is even it follows in the same way that

$$f(z + m/2) = -f(z). \tag{14}$$

It is easily seen by a simple contradiction that $p = m, q = m/2$ are the least positive integers such that

$$f(z + p) = f(z), \quad f(z + q) = -f(z).$$

Hence m is the true period in (13), and $m/2$ the true half period in (14).

CALIFORNIA INSTITUTE OF TECHNOLOGY

NOTE ON THE CONVERGENCE OF A SEQUENCE OF APPROXIMATING POLYNOMIALS*

BY DUNHAM JACKSON

Various results have been published, by the present writer and others, with regard to the convergence of sequences of approximating polynomials defined by minimizing an integral of the form

$$\int_a^b \rho(x) \mid f(x) - P_n(x) \mid^m dx, \tag{1}$$

in which $f(x)$ is a function defined and continuous for $a \leqq x \leqq b$, and subject to such further hypotheses as the particular convergence proof in question demands, $\rho(x)$ is summable and non-negative over the interval, m is a given positive number >1, and $P_n(x)$ is a polynomial of the nth degree (this expression being understood throughout to mean a polynomial of the nth degree *at most*). The proof of convergence as n becomes infinite is most easily obtained if $\rho(x)$ has a positive lower bound, at least over an interval containing the point at which convergence is to be shown.† The author has briefly discussed for the corresponding trigonometric case the question of convergence at a point at which the weight function vanishes, under certain restrictive hypotheses as to the manner of vanishing.‡ The purpose of this note is to point out that if $f(x)$ is *analytic* in a sufficiently extended region of the complex plane the conditions relative to the vanishing of $\rho(x)$ can be greatly generalized. *It is sufficient that* $\rho(x)$ (supposed summable and non-negative over the interval of integration) *be different from zero over a set of positive measure*, or in other words that

$$\int_a^b \rho(x)dx > 0.$$

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* Presented to the Society, September 9, 1930.

† See, for example, D. Jackson, *The Theory of Approximation*, New York, 1930, pp. 96–101.

‡ *A generalized problem in weighted approximation*, Transactions of this Society, vol. 26 (1924), pp. 133–154; see pp. 153–154. See also the paper by Shohat in the Mathematische Annalen cited below.

An existence proof for the approximating polynomial can be given without difficulty. Close restrictions on $f(x)$ are irrelevant for this purpose; it is sufficient that $f(x)$ be bounded and measurable. The exponent m furthermore may be any positive number, not necessarily greater than unity. A discussion by the author for the case of trigonometric approximation* can be adapted immediately to prove the existence of a minimizing polynomial in the present problem, if $\rho(x)$ is bounded and measurable. It remains to show that the conclusion holds whenever $\rho(x)$ is summable.†

The transition is almost immediate. Whether ρ is bounded or not, the value of the integral (1) is a continuous function of the coefficients in $P_n(x)$. For bounded ρ, the existence proof consists essentially in showing that if an upper bound is assigned for the value of the integral, an upper bound is thereby imposed on the coefficients, so that the problem is solved by the theorem on the minimum of a continuous function in a closed region. If ρ is unbounded, let N be a positive number such that $0<\rho(x)\leqq N$ over a point set of positive measure; the fact that there is such an N is assured by the hypotheses that have been laid down. Let a function $\rho_N(x)$ be defined as equal to $\rho(x)$ at points where $\rho\leqq N$, and let $\rho_N(x)=N$ where $\rho\geqq N$. If an upper bound is assigned for the integral (1), the corresponding integral with ρ replaced by ρ_N can not exceed this upper bound, since $\rho_N\leqq\rho$ everywhere, and application of the previous reasoning to the integral with the bounded function ρ_N yields an upper bound for the coefficients in $P_n(x)$. The proof is then completed as before.

Under the assumption that $m>1$, the polynomial reducing the integral to its minimum is uniquely determined for each value of n. This may be deduced, as in similar situations elsewhere, from the observation that if two different polynomials give the integral the same value their average gives it a smaller one.

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* Transactions of this Society, loc. cit., pp. 133–139.

† A somewhat different proof for the case $m\geqq 1$ is given by J. Shohat, *On the polynomial and trigonometric approximation of measurable bounded functions on a finite interval*, Mathematische Annalen, vol. 102 (1929), pp. 157–175; pp. 160–161. See also J. Shohat, *On the polynomial of the best approximation to a given continuous function*, this Bulletin, vol. 31 (1925), pp. 509–514.

Let it be supposed now that $f(x)$ is continuous for $a \leqq x \leqq b$. (This hypothesis will presently be replaced by a still more restrictive one.) For any specified $m > 1$, and for each value of n, let $P_n(x)$ be the particular polynomial of the nth degree which minimizes the integral (1). As a general notation, let the symbol $[y]_{m-1}$ represent the value 0 if $y = 0$, and the quantity $|y|^m/y$ if $y \neq 0$, so that $[y]_{m-1}$ has the absolute value of $|y|^{m-1}$ and the algebraic sign of y itself. Then, if $Q_n(x)$ is any polynomial of the nth degree,

$$\int_a^b \rho(x)[f(x) - P_n(x)]_{m-1} Q_n(x)dx = 0. \tag{2}$$

The proof is an immediate generalization of one given by the writer elsewhere.* The generalization consists in the admission of the weight function $\rho(x)$, which was not present in the earlier discussion cited. Its presence, however, makes no difference in the argument, if it is noted that the elementary sufficient condition for differentiating under the sign of integration can be replaced by the assertion that

$$\frac{d}{dy}\int_a^b \rho(x)\,\phi(x, y)dx = \int_a^b \rho(x)\,\phi_y(x, y)dx$$

provided that ϕ and ϕ_y are continuous functions of both variables and $\rho(x)$ is summable.

This being granted, the reasoning proceeds as follows. If $f(x) - P_n(x)$ does not change sign at least $n+1$ times in (a, b), a polynomial $Q_n(x)$ of the nth degree can be constructed so as to have the same sign as $f(x) - P_n(x)$ at all points where $f(x) - P_n(x) \neq 0$. The integrand in (2) then is positive or zero over the whole interval, and positive at all points where $\rho(x)$ and $f(x) - P_n(x)$ are both different from zero. Since the value of the integral must be zero, the integrand must vanish almost everywhere, and $f(x) - P_n(x)$ consequently must vanish wherever $\rho(x) \neq 0$, with the exception of a set of zero measure at most. As $\rho(x)$ by hypothesis is positive over a set of positive measure, *$f(x) - P_n(x)$ must vanish throughout a set of positive measure, or else change sign at least $n+1$ times in the interval.*

* *On functions of closest approximation*, Transactions of this Society, vol. 22 (1921), pp. 117–128; pp. 122–124.

Let it be assumed finally that $f(x)$ is analytic for $a \leqq x \leqq b$, and analytic furthermore throughout a circle of the complex plane having its center at the middle point of the interval, and a radius greater than three times the half-length of the interval. If $P_n(x)$ is equal to $f(x)$ at all points of a set of positive measure, it is equal to $f(x)$ identically; in other words, $f(x)$ is itself a polynomial of the nth degree, and is its own polynomial of best approximation, of the nth or any higher degree. In the contrary case, $P_n(x)$ is equal to $f(x)$ for at least $n+1$ distinct values of x in (a, b). This condition being fulfilled for each value of n, a well known argument* shows that $P_n(x)$ *converges uniformly toward* $f(x)$ *throughout a circle of the complex plane containing the interval* (a, b) *in its interior.*† The hypothesis that $f(x)$ is analytic throughout a smaller circle gives convergence in a correspondingly more restricted region.

The method is applicable also if the approximating polynomial is subjected to auxiliary conditions, of the sort discussed by the author in a previous paper,‡ the weight function retaining the degree of generality admitted here.

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* See, for example, D. Jackson, *On the approximate representation of analytic functions*, this Bulletin, vol. 34 (1928), pp. 56–62, and references there to Hermite and Runge.

† By a slight modification, the convergence proof can be made to apply for $0 < m \leqq 1$ also. For in any case (the function $f(x)$ being analytic) the *multiplicities* of the roots of the remainder $f(x) - P_n(x)$ on (a, b) must add up to at least $n+1$, if $P_n(x)$ is any polynomial reducing the integral (1) to its minimum value, since otherwise a polynomial $Q_n(x)$, of the nth degree at most, could be constructed so as to have roots at the same points with the same multiplicities, and the same sign as the remainder where the remainder is different from zero; and then $[f(x) - P_n(x)]/Q_n(x)$, defined by continuity at the points where numerator and denominator vanish, would be positive over the whole interval and have a positive minimum h there; and the subtraction of $hQ_n(x)$ would diminish the magnitude of the remainder at all points where it does not vanish, and would diminish the value of the integral. The count of multiplicities being sufficient for the proof of convergence, the conclusion is obtained as before. (Note added October 17, 1930.)

‡ This Bulletin, loc. cit.

ON A FUNCTION CONNECTED WITH A CUBIC FIELD*

BY LEONARD CARLITZ†

By examining the functional equation of the zeta function and similar functions of an algebraic field, Hecke‡ has indicated a mode of constructing three types of modular forms. The first two types, associated with the rational Dirichlet L-functions and the Dedekind zeta function in an imaginary quadratic field, respectively, were already known. The third type, associated with a real quadratic field, was new.

Following Hecke, we construct in this note a certain function associated with a cubic field of negative discriminant. Let K denote such a field, and let $\zeta_K(s)$ be the zeta function in this field. Then as Artin§ has shown, $\zeta_K(s)/\zeta(s)$, the quotient of this function by the Riemann zeta, is an entire function of s. Indeed

$$\zeta_K = \zeta(L_1L_2)^{1/2}, \tag{1}$$

where L_1 and L_2 are L-functions in the imaginary quadratic field generated by the square root of d, the discriminant of K.

If we define $G(n)$ by

$$\frac{\zeta_K}{\zeta}(s) = \sum_{n=1}^{\infty} \frac{G(n)}{n^s}, \quad \sigma = R(s) > 1,$$

then the function we shall consider may be exhibited as

$$M(y) = \sum_{1}^{\infty} G(n)e^{2n\pi yi/\Delta},$$

$$I(y) > 0, \quad \Delta = |d|^{1/2} > 0.$$

Using a well known formula of Mellin,‖

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† National Research Fellow in Mathematics.

‡ *Zur Theorie der elliptischen Modulfunktionen*, Mathematische Annalen, vol. 97 (1926), pp. 210–242.

§ *Über die Zetafunktionen gewisser algebräischen Zahlkörper*, Mathematische Annalen, vol. 89 (1923), pp. 147–156.

‖ Mellin, *Abriss einer einheitlichen Theorie der Gamma- und hypergeometrischen Funktionen*, Mathematische Annalen, vol. 68 (1910), pp. 305–337.

$$e^{2n\pi yi/\Delta} = \frac{1}{2\pi i}\int_{3/2-\infty i}^{3/2+\infty i}\left(\frac{-2n\pi yi}{\Delta}\right)^{-s}\Gamma(s)ds,$$

$$(2)\qquad \begin{cases} M(y) = \dfrac{1}{2\pi i}\displaystyle\sum_1^\infty \int\left(\frac{-2\pi yi}{\Delta}\right)^{-s}\Gamma(s)\frac{G(n)}{n^s}ds \\ \qquad = \dfrac{1}{2\pi i}\displaystyle\int\left(\frac{-2\pi yi}{\Delta}\right)^{-s}\Gamma(s)\frac{\zeta_K}{\zeta}(s)ds, \end{cases}$$

the interchange of integration and summation is easily justified. We now consider the integral of

$$\left(\frac{-2\pi yi}{\Delta}\right)^{-s}\Gamma(s)\frac{\zeta_K}{\zeta}(s)$$

taken in the positive direction around the rectangle of vertices

$$\frac{3}{2} \pm Ai, \quad -\frac{1}{2} \pm Ai, \quad A > 0.$$

Since ζ_K/ζ is integral and since ζ_K has a zero* at $s=0$, the integrand is regular within and on the boundary of the rectangle. Furthermore, using (1) and†

$$L_i(s) = O(t^2), \qquad (i = 1, 2),$$

$$s = \sigma + it, \quad \sigma \geqq -\frac{1}{2};$$

$$\frac{\zeta_K}{\zeta}(s) = O(t^2) \text{ for } \sigma \geqq -\frac{1}{2};$$

we see that the integrals along the vertical boundaries will converge and those along the horizontal boundaries will approach zero when A becomes infinite.‡ Therefore

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* Landau, *Einführung in die elementare und analytische Theorie der algebraischen Zahlen und der Ideale*, 1918, Satz 155.

† Landau, *Ueber Ideale und Primideale in Idealklassen*, Mathematische Zeitschrift, vol. 2 (1918), p. 106.

‡ See Landau, *Vorlesungen über Zahlentheorie*, vol. 1, 1927, p. 215.

$$\int_{3/2-\infty i}^{3/2+\infty i} \left(\frac{-2\pi y i}{\Delta}\right)^{-s} \Gamma(s)\frac{\zeta_K}{\zeta}(s)ds$$

$$= \int_{-1/2-\infty i}^{-1/2+\infty i} \left(\frac{-2\pi y i}{\Delta}\right)^{-s} \Gamma(s)\frac{\zeta_K}{\zeta}(s)ds$$

(3)
$$= \int_{3/2-\infty i}^{3/2+\infty i} \left(\frac{-2\pi y i}{\Delta}\right)^{s-1} \Gamma(1-s)\frac{\zeta_K}{\zeta}(1-s)ds.$$

Now

$$\zeta_K(1-s) = \left(\frac{\Delta}{2\pi^{3/2}}\right)^{2s-1} \frac{\Gamma\left(\frac{s}{2}\right)\Gamma(s)}{\Gamma\left(\frac{1-s}{2}\right)\Gamma(1-s)}\zeta_K(s),$$

and

$$\zeta(1-s) = \pi^{-(2s-1)/2}\frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)}\zeta(s),$$

so that

$$\frac{\zeta_K}{\zeta}(1-s) = \left(\frac{\Delta}{2\pi}\right)^{2s-1}\frac{\Gamma(s)}{\Gamma(1-s)}\frac{\zeta_K}{\zeta}(s).$$

Substituting in (3), we find that

$$M(y) = \frac{1}{2\pi i}\int_{3/2-\infty i}^{3/2+\infty i}\left(\frac{-2\pi y i}{\Delta}\right)^{s-1}\left(\frac{\Delta}{2\pi}\right)^{2s-1}\Gamma(s)\frac{\zeta_K}{\zeta}(s)ds$$

$$= \frac{1}{-yi}\frac{1}{2\pi i}\int_{3/2-\infty i}^{3/2+\infty i}\left(\frac{2\pi i}{y\Delta}\right)^{-s}\Gamma(s)\frac{\zeta_K}{\zeta}(s)ds.$$

Comparing this with (2), we see at once that

$$M(y) = -\frac{1}{yi}M\left(-\frac{1}{y}\right),$$

which is the property of $M(y)$ sought.

CALIFORNIA INSTITUTE OF TECHNOLOGY

PENOSCULATING CONICS OF A PLANE CURVE*

BY J. W. LASLEY, JR.

1. *The Penosculating Parabolas.* There is a one-parameter family of parabolas on three non-collinear points of a plane curve. If two of these three points approach a third point at which the tangent to the given curve is well-defined and the curvature is definitely non-vanishing, there is determined a one-parameter family of penosculating parabolas at the point. Abel Transon† found that the directrices of these parabolas all pass through a point on the normal to the curve at the given point P on the convex side of the curve whose distance from P is equal to one-half the radius of curvature. He found that the locus of the foci of these parabolas is a circle whose diameter is that half of the radius of curvature that terminates at P. Transon obtained a construction for one of the parabolas, the osculating parabola. We shall now generalize this construction so as to make it applicable to any parabola of the family.

For this purpose we shall associate with each parabola of the pencil a line on P, parallel to the axis of the parabola. We shall call this line the axis parallel, and the pencil of lines to which it belongs the axis pencil. To each line of the axis pencil there corresponds a unique parabola of the pencil of penosculating parabolas except in the case of the tangent to the given curve. We shall make correspond to this tangent the degenerate parabola consisting of that tangent counted twice. In this way we have set up a one-to-one correspondence between the parabolas of the pencil and the axis parallels. In this correspondence there corresponds to the normal as an axis parallel a parabola whose axis is that normal, whose tangent at its vertex is the tangent to the given curve, and whose focus is the center of the circular locus of the orthogonal projections of the center of curvature on the axis parallels.

To construct a parabola of the pencil of penosculating parabolas we proceed as follows. Project orthogonally the center of curvature C of the given curve at P upon the axis parallel a cor-

* Presented to the Society, February 28, 1931.

† Journal de Mathématiques, vol. 6 (1841), p. 191.

responding to this parabola, obtaining the point D. Project orthogonally D upon the normal at P into a point E. The line on E parallel to a is the axis of the required parabola. The focus of the parabola will lie on a line s through the point P which makes with the normal an angle equal to that made by the axis parallel a, but on the opposite side. This line s we shall call the symmetric line for the axis parallel. The focus is then the point of intersection F of the axis of the parabola and the symmetric line. Let A be the point on the normal at P on the convex side of the curve whose distance from P is equal to one-half the radius of curvature. The directrix of the parabola is the line on A perpendicular to the axis of the parabola. The focus and directrix of the penosculating parabola having been located, its construction may now be regarded as complete. Of course, the locus of intersections of the axis parallels and corresponding directrices is a circle tangent to the given curve at P. This circle has a diameter equal to one-half the radius of curvature, and is consequently the reflexion in the tangent of the circular locus of foci of penosculating parabolas. If in the foregoing construction for any penosculating parabola the axis parallel is taken to be the axis of aberrancy of Transon, we obtain the osculating parabola. That the construction can be generalized by having the axis parallel play the role of the axis of aberrancy seems not to have been noticed.

2. *The Penosculating Equilateral Hyperbolas.* There is also at P a one-parameter family of penosculating equilateral hyperbolas obtained from the pencil of equilateral hyperbolas on three non-collinear points of a given curve by letting two of the three points approach the third. These hyperbolas seem to have received practically no attention. Cesàro* obtained for one of them, the osculating equilateral hyperbola, a canonical form for its equation referred to a local reference frame. We shall extend the correspondence outlined above so as to include these hyperbolas and give a construction available for any penosculating equilateral hyperbola of the pencil.

The locus of the centers of these hyperbolas is a circle on the convex side of the given curve with diameter equal to the radius of curvature. This circle touches the given curve at P and has,

* *Lezioni di Geometria Intrinseca*, Naples, 1896, p. 59.

then, the same normal there. It is the reflexion in the tangent of the circular locus of projections of the center of curvature on the axis parallels. Moreover, the center of this circle is the vertex of the pencil of directrices of penosculating parabolas mentioned by Transon. We shall make use of this circle to extend our correspondence to include the hyperbolas. For this purpose we make correspond to each axis parallel that hyperbola whose center is the further intersection of the axis parallel with the circular locus of centers of penosculating equilateral hyperbolas. To each hyperbola there is a unique axis parallel, namely that determined by the center of the hyperbola and P. In this correspondence we shall assign to the tangent as an axis parallel the degenerate hyperbola consisting of the tangent and the normal to the given curve at P. To the normal there will correspond a hyperbola whose transverse axis is the normal and whose conjugate axis is parallel to the tangent and distant from it on the convex side of the curve an amount equal to the radius of curvature. For this hyperbola we have a very simple construction for obtaining its asymptotes and axes. With P as a center and with the radius of curvature as radius describe a circle. This circle meets the tangent in points which joined to P give the asymptotes of the hyperbola; it meets the normal in points which joined to P give the axes of the hyperbola. Of course, the above refers only to the hyperbola corresponding to the normal, but it will be seen below that it is a special case of a more general construction available for all penosculating equilateral hyperbolas.

To construct any hyperbola of the pencil we proceed as follows. With P as a center and with a radius equal to the projection of the radius of curvature on the corresponding axis parallel describe a circle. This circle passes through the center of the required hyperbola as picked out by the further intersection of the axis parallel with the locus of centers. This circle meets the tangent to the given curve at P in points which joined to the center give the asymptotes; it meets the normal in points which joined to the center give the axes. Since we know that the required hyperbola goes through P, and now have a construction for its asymptotes, we may consider the construction for any penosculating equilateral hyperbola complete. It is interesting to note that for every hyperbola of the pencil it is so that the

projection of the radius of curvature of the given curve upon the axis parallel is a mean proportional between the intercepts of the tangent of the hyperbola on the tangent. If, in particular, in the foregoing construction, the chosen axis parallel is the axis of aberrancy of Transon, the hyperbola obtained is the osculating equilateral hyperbola of Cesàro. Our correspondence, then, makes correspond the osculating parabola, the axis of aberrancy and the osculating equilateral hyperbola.

3. *Transon's Parabola.* There is at an ordinary point of a curve a one-parameter family of penosculating conics obtained from the pencil of conics on four points by letting three of the points approach the fourth. Wilczynski* has pointed out that among these conics there will ordinarily be one, and only one, parabola, the osculating parabola. There will also be one, and only one, equilateral hyperbola, the osculating equilateral hyperbola. Wilczynski notes, too, that there will be one, and just one, conic for which the order of contact at the point will rise to four. This conic he calls the osculating conic. Transon found that the locus of centers of penosculating conics is the axis of aberrancy. He found also that the axes of these penosculating conics envelop a parabola, which we shall call, with Wilczynski, Transon's parabola. Due to Transon also is the fact that the focus of this parabola is the orthogonal projection G of the center of curvature of the given curve upon the line joining P to the focus of the osculating parabola, that is, upon the line we have called the symmetric line for the axis of aberrancy. In the terminology we are employing here this focus appears also as the point of intersection of the symmetric line for the axis of aberrancy and the circular locus of feet of perpendiculars let fall from the center of curvature on the axis parallels. It is, too, the reflexion in the normal of the orthogonal projection of the center of curvature upon the axis of aberrancy. Transon found that the directrix of this parabola is the axis of aberrancy. Consequently the axis of the parabola is the line on G perpendicular to the axis of aberrancy. It is of interest, though not noted by Transon, that the focus of the osculating parabola lies on the symmetric line for the axis of aberrancy half way between the given point P and the focus of Transon's parabola. The tangent

* This Bulletin, vol. 22, p. 317.

to Transon's parabola at the vertex always coincides with the axis of the osculating parabola. The tangent at the vertex of the osculating parabola is always parallel to the axis of Transon's parabola, but coincides with it if, and only if, the given curve is a logarithmic spiral whose tangents cut the radii vectores at an angle whose cotangent is $\pm 3\sqrt{2}$. The directrix of the osculating parabola coincides with the axis of Transon's parabola if, and only if, the given curve is a logarithmic spiral with an angle whose cotangent is $\pm 3\sqrt{3}$. The symmetric line for the axis of aberrancy is usually quite different from the axis of Transon's parabola. It coincides with it if, and only if, the given curve is a logarithmic spiral whose angle has ± 3 for its cotangent. In this event the aberrancy of the given curve is unity at the point, the vertex of Transon's parabola coincides with the focus of the osculating parabola and Transon's parabola touches the tangent and the normal of the given curve at the ends of the latus rectum. Wilczynski noted that the tangent and the normal of the given curve always touch Transon's parabola. It may be added that the normal always touches it at the center of curvature, and that a line joining the center of curvature to the focus of the osculating parabola always meets the tangent to the given curve in a point T at which Transon's parabola touches that tangent. The reflexion J of the center of curvature C in the tangent when joined to T gives that diameter of Transon's parabola which bisects all chords parallel to the tangent.

4. *Construction for a Penosculating Conic.* By means of Transon's parabola we can now obtain a construction for any penosculating conic. Wilczynski noted for the osculating conic a property true for all penosculating conics, namely, that the tangents to Transon's parabola from a point on the axis of aberrancy are the axes of the conic which has that point as center. It turns out that one of these axes always touches Transon's parabola between the points at which that parabola touches the tangent and the normal to the given curve. This axis is the axis on which lie the foci of the conic. If now a circle be described upon that segment of the other axis included between its intersections with the tangent and the normal of the given curve, this circle will meet the first named axis in the foci of the desired penosculating conic. Since this conic is known to pass through

the point P on the given curve, we now know the foci and one point, and the conic can readily be drawn. With the exception of the osculating parabola, this construction applies to any member of the pencil of penosculating conics. Since a construction for the osculating parabola has been outlined above, we may regard the problem of constructing penosculating conics as completely solved.

5. *The Ellipse of Minimum Eccentricity.* Wilczynski showed that among the conics just considered there is an ellipse of minimum eccentricity. We can now obtain the following construction for this ellipse. Let us project orthogonally the center of curvature of the given curve upon the axis of aberrancy. This projection is the center of the ellipse of minimum eccentricity. With the position of the center known we can construct the ellipse by means of the construction just outlined for penosculating conics in general. It is interesting to note that a circle with center at P and with radius equal to the semi-diameter of this ellipse which terminates at P meets the axis of aberrancy again in the center of the osculating equilateral hyperbola, and meets the symmetric line for the axis of aberrancy on the concave side of the given curve in the focus of Transon's parabola. Penosculating ellipses of equal eccentricity occur in pairs. Wilczynski found that the centers O_1 and O_2 of any two of these ellipses of equal eccentricity are so situated on the axis of aberrancy that if O denotes the center of the ellipse of minimum eccentricity, PO is the geometric mean between PO_1 and PO_2. That is, the centers of the ellipses of equal eccentricity form an involution whose center is the center of the ellipse of minimum eccentricity. Although for a given suitable value for the eccentricity there are two ellipses that have that eccentricity, this is not true of the penosculating hyperbolas, if we confine our attention to the eccentricities corresponding to the real foci and real directrices. Although an ellipse has only one real eccentricity, the hyperbola ordinarily has two. For a given suitable value for the eccentricity of an hyperbola there are two penosculating hyperbolas, one with this eccentricity for its real foci and real directrices, the other with this eccentricity for its imaginary foci and imaginary directrices. If now we associate in pairs those hyperbolas which have equal eccentricities, we can extend the

above involution to the entire family of penosculating conics. If, further, we define the cross-ratio of four members of a one-parameter family of conics to be that of the parameters which single them out in the family, we can conclude that the pairs of conics of equal eccentricity separate harmonically the osculating equilateral hyperbola and the ellipse of minimum eccentricity. In this involution of penosculating conics so determined the partner of the osculating parabola is the degenerate parabola consisting of the tangent to the given curve counted twice. Although as Wilczynski found, there is in this family a unique ellipse of minimum eccentricity, there is no hyperbola of maximum eccentricity. The osculating equilateral hyperbola and the ellipse of minimum eccentricity appear as the double conics in this involution of penosculating conics.

THE UNIVERSITY OF NORTH CAROLINA

THE BITANGENTIAL CURVE*

BY T. R. HOLLCROFT

1. *Introduction.* The bitangential curve of an algebraic surface is the locus of the points of contact of planes which touch the surface in two points. It is the reciprocal of the nodal developable. The order of this curve, for an algebraic surface without singularities of any kind, has been determined by Cayley,† who called it the node-couple curve.

When the surface has a nodal and a cuspidal curve, each of given order, the orders of the spinodal and flecnodal curves were found by Cayley.‡ The order of the bitangential curve of a surface with nodal and cuspidal curves, however, has not been found explicitly. Basset§ makes the following statement: "I have not succeeded in ascertaining the reduction in the degree of the bitangential curve which is produced by a nodal and a cuspidal curve; but if the reduction is denoted by $xb+yc$, the method of the preceding paragraph indicates that x and y are

* Presented to the Society, December 30, 1930.

† A. Cayley, Collected Mathematical Papers, vol. 6, p. 346.

‡ A. Cayley, loc. cit., vol. 6, pp. 342–343.

§ A. B. Basset, *A Treatise on the Geometry of Surfaces*, Cambridge, 1910, p. 280.

functions of the degree n of the surface," which, together with a similar statement in §59, page 40, indicates that the order of the bitangential curve under these conditions has not been determined.

The purpose of this paper is to determine the order of the bitangential curve of an algebraic surface that has nodal and cuspidal curves and also singular points and planes. The necessary formulas for this purpose have all been derived by Cayley and Zeuthen, so that the work in this paper consists merely in collecting and combining these formulas.

2. *Notation.* The following symbols will be used:

$n[n']$ order [class] of algebraic surface f.

a' class of plane section of f.

κ' number of inflections of plane section of f.

$b[c]$ order of nodal [cuspidal] curve of f.

$k[h]$ number of apparent double points of nodal [cuspidal] curve.

t number of triple points of nodal curve.

$\beta[\gamma]\{i\}$ number of intersections of nodal and cuspidal curves which are cusps on cuspidal [nodal] {neither} curve.

$\rho'[\sigma']$ order of bitangential [spinodal] curve of f.

The above notation is that used by Cayley and Salmon. The complete list (given in the references*) of symbols representing all the possible singularities of an algebraic surface is very long and in a short paper, it seems unnecessary to define all of them.

3. *Order of the Bitangential Curve.* The following formulas derived by Cayley† hold for an algebraic surface with no further singularities than a nodal and a cuspidal curve:

$$a'(n' - 2) = \kappa' - \rho' + 2\sigma', \tag{1}$$

$$a' = n(n - 1) - 2b - 3c, \tag{2}$$

$$\kappa' = 3n(n - 2) - 6b - 8c, \tag{3}$$

* Salmon-Rogers, *A Treatise on the Analytical Geometry of Three Dimensions*, 5th ed., 1915, vol. 2, pp. 314–318. H. G. Zeuthen, *Sur la théorie des surfaces réciproques*, Mathematische Annalen, vol. 10 (1876), pp. 450–454.

† A. Cayley, loc. cit., vol. 11, pp. 228–231.

$$(4) \qquad n' = n(n-1)^2 - n(7b + 12c) + 4b^2 + 8b + 9c^2 + 15c - 8k - 18h - 9t + 18\beta + 12\gamma + 12i,$$

$$(5) \qquad \sigma' = 4n(n-2) - 8b - 11c.$$

Solving (1) for ρ' and substituting in it the values of a', κ', n', σ' as given in formulas (2), (3), (4), (5), we find

$$\begin{aligned}\rho' = {} & n(n-2)(n^3 - n^2 + n - 12) - n(n-1)[3n(3b + 5c) \\ & - 2b(2b + 5) - 9c(c + 2)] + n(2b + 3c)(7b + 12c) \\ & - (2b + 3c)[4(b-1)(b+3) + 3c(3c + 5)] + 2b \\ & - [n(n-1) - 2b - 3c][18(h - \beta) - 12(i + \gamma) + 8k + 9t].\end{aligned}$$

This is the order of the bitangential curve when the surface has a nodal curve of order b and a cuspidal curve of order c. The first product is the order of the bitangential curve for a non-singular surface. The reduction in the order of the bitangential curve due to the nodal and cuspidal curves is the sum (with the signs changed) of all the terms of the above expression after the first product.

The order can also be found by the same method when the surface has any of the point or plane singularities defined in the complete list of singularities. In this case, numbers associated with certain of these singularities occur in formulas (1), (4) and (5) as determined by Zeuthen in the reference cited. The formula for ρ' is very long when all possible singularities are included and, therefore, will not be given here. The reduction in the order of the bitangential curve due to any given singularity of the surface can be readily determined, however, as above.

WELLS COLLEGE

FUNCTIONAL EQUATIONS FOR TOTIENTS*

BY E. T. BELL

1. *Totient Functions.* Let $p_1, \cdots, p_a$ be the distinct prime factors of n. The number of different sets of k equal or distinct positive integers less than or equal to n whose G. C. D. is prime to n is the Jordan totient $\phi_k(n)$ of order k, and

$$\phi_k(n) \equiv n^k(1 - p_1^{-k}) \cdots (1 - p_a^{-k}). \tag{1}$$

If k is complex, $\phi_k(n)$ is defined by (1). The special case $k=1$ gives Euler's $\phi(n)$. The case $k=0$ is trivial and will be ignored.

We say that the numerical function $f \equiv f(n)$ is *factorable* if $f(1)=1$ (which is adjoined to the definition of $f(n)$ if $f(n)$ is defined arithmetically for $n>1$ but not for $n=1$), and if $f(mn) = f(m)\,f(n)$ for all pairs of coprime integers m, n.

Factorability is distinct from separability, which we define as follows. Let α denote a variable integer >0 and p a fixed but arbitrary prime. Write $p^\alpha \equiv y$, $p \equiv x$, and regard x, y as independent variables. Then the factorable numerical function f is *separable* if there exist numerical functions g, h such that $f(p^\alpha) \equiv g(x)\,h(y)$. For example, ϕ_k is separable; σ_k, where $\sigma_k(n)$ is the sum of the kth powers of all the divisors of n, is not.

If f is separable, say $f(p^\alpha) \equiv g(x)\,h(y)$, and if further $h(y)$ is of the form $\sum_{i=1}^{s} (a_i y^i + b_i y^{-i})$, where the a_i, b_i are constants and $a_s b_s \neq 0$, we say that f is *simply* separable of *extent* s. Thus ϕ_k is simply separable of extent 1. *Regarding $f(p^\alpha)$ as a function of x and y, we shall write $f(p^\alpha) \equiv f(x, y)$.*

It is well known that ϕ_k is the unique solution f of the functional equation

$$\sum_{d|n} f(d) = n^k, \qquad (n = 1, 2, \cdots). \tag{2}$$

Although a proof of this will not be required, we give one to contrast the algebra involved with another, which will be used.

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* Presented to the Society, November 29, 1930.

Write $u_r(n) \equiv n^r$ for all integers $n > 0$. Then* (2) is $u_0 f = u_k$, which is linear in f and hence has the unique solution $f = u_k u_0^{-1} = u_k \mu$ ($\mu \equiv$ Möbius' function). Hence, by (1), $f = \phi_k$ is the unique solution of (2).

When k is a positive rational integer, (2) can be obtained at once from the arithmetical definition of $\phi_k(n)$, and in fact one of the usual ways of deriving the properties of Euler's $\phi(n)$ ($k = 1$) is by first proving (2) directly from the definition of $\phi(n)$. Another functional equation for ϕ_k, stated in (3), of which ϕ_k is again the unique solution, conceals some more recondite arithmetical property of totients. In itself it is remarkable enough to merit independent notice. It also gives an example in a new algebra, devised by D. H. Lehmer, which is isomorphic with common algebra (the theory of an abstract infinite field), and which reveals interesting new aspects of numerical functions radically different from those depending ultimately upon Dirichlet multiplication, as was the case with the algebra cited in the preceding footnote. In this algebra, since it is isomorphic with a field, multiplication has a unique inverse and a unique identity element; multiplication and division are the operations of greatest arithmetical interest. Applied to simply separable functions the new multiplication gives the second characteristic equation (3) for ϕ_k. A third characteristic equation is given in §3.

All that will be required of Lehmer's algebra is the definition of multiplication. Let m, n, r be positive integers. The number of sets (m, n) such that the L. C. M. of m, n has the constant value r is finite. Let f, g be numerical functions, and let the summation refer to all of the pairs (m, n) just defined. Then $\sum f(m)g(n)$ is a numerical function of r, say $h(r)$, and

$$\sum f(m)g(n) = h(r), \qquad (r = 1, 2, \cdots),$$

is written $(fg) = h$, which defines the *product* (fg). As stated, the multiplication (fg) is associative and commutative, and has a unique inverse. The last implies that if f is any numerical function other than the identically zero function, there exists

* For the algebra (symbolic method) used, see *Outline of a theory*, etc., Journal Indian Mathematical Society, vol. 17 (1928), where references to previous papers are given. If f, g are numerical functions, $f = g$ means that $f(n) = g(n)$, $(n = 1, 2, \cdots)$.

a *unique* numerical function u (the same for all f) such that $(fg)=u$ has a unique solution g.

We write $(ff)\equiv(f^2)$, $(f^2)(n)\equiv f^2(n)$. If f is separable, say $f(p^\alpha)\equiv f(x,y)\equiv g(x)\,h(y)$, and if $n=p_1^{\alpha_1}\cdots p_a^{\alpha_a}$, where $p_1, \cdots, p_a$ are distinct primes, then

$$f(n) = f(p_1^{\alpha_1})\cdots f(p_a^{\alpha_a}) \equiv g(x_1)\cdots g(x_a)h(y_1)\cdots h(y_a),$$

where $x_i\equiv p_i$, $y_i\equiv p_i^{\alpha_i}$. Hence it is sufficient to discuss the equation $f(x,y)=g(x)\,h(y)$.

If f is separable, f^2 in general is not separable. In §2 we determine all simply separable f such that f^2 is separable. The problem of determining *all* separable f (not merely *simply* separable f) such that f^2 is separable, leads to a functional equation which appears to be quite intractable. When f is simply separable and f^2 is separable, we shall see that f^2 also is simply separable.

If both f, f^2 are separable, say

$$f(x,y)\equiv g(x)h(y),\quad f^2(x,y)\equiv G(x)H(y),$$

then also (k is any complex number $\neq 0$),

$$f(x^k,y^k)\equiv g(x^k)h(y^k),\quad f^2(x^k,y^k)\equiv G(x^k)H(y^k),$$

and conversely this implies the preceding. It is sufficient therefore to consider the case $k=1$. The second characteristic equation of ϕ_k is given by the next theorem.

THEOREM 1. *The unique simply separable solution $f(x,y)$ of*

$$f^2(x,y) = f(x^2,y^2) \tag{3}$$

is $f=\phi_k$, where k is an arbitrary complex number $\neq 0$.

This theorem, proved in §2, end, originated in an attempt to relate the following astonishing property of $(\phi_k\phi_l)$ to Jordan's original definition when k, l are positive integers: if k, l are any complex numbers other than zero, $(\phi_k\phi_l)=\phi_{k+l}$. This is due to von Sterneck.* It is shown in §3 that this property gives a third functional equation for totients.

2. *Simply Separable f and f^2.* As in §1, write $p\equiv x$, $p^\alpha\equiv y$ (p

* Monatshefte für Mathematik, vol. 5 (1894), pp. 255–266.

prime, α an arbitrary positive integer), and define $Q, \cdots, T_j$ by

$$Q(y) \equiv \sum_{j=1}^{s}(a_j y^j + b_j y^{-j}), \qquad a_s b_s \neq 0;$$

$$A(x) \equiv \prod_{j=1}^{s}(x^j - 1), \quad A_j(x) \equiv A(x)/(x^j - 1);$$

$$R_j(x) \equiv -b_{s-j}(x^{s-j} + 1)A_{s-j}(x), \quad (j = 0, 1, \cdots, s-1);$$

$$S(x) \equiv A(x) + P(x)\sum_{j=1}^{s}(b_j - a_j x^j)A_j(x);$$

$$T_j(x) \equiv a_j(x^j + 1)A_j(x), \qquad (j = 1, \cdots, s).$$

Then, if the function $f(x, y)$ is simply separable of extent s, and $f(x, y) \equiv P(x)Q(y)$, we have

$$f^2(x, y) = \frac{P(x)Q(y)}{A(x)y^s}\left[P(x)\sum_{j=0}^{s-1}R_j(x)y^j + 2S(x)y^s + P(x)\sum_{j=1}^{s}T_j(x)y^{s+j}\right].$$

For, from the definition of Lehmer's product (hg) in terms of L. C. M., where h, g are any numerical functions,

$$(hg)(p^\alpha) = h(p^\alpha)\left[1 + \sum_{j=1}^{\alpha}g(p^j)\right] + g(p^\alpha)\left[1 + \sum_{j=1}^{\alpha}h(p^j)\right] - h(p^\alpha)g(p^\alpha).$$

In this take $h = g = f$, $f(x, y) \equiv P(x)\ Q(y)$. Then, by a short reduction,

$$f^2(x, y) \equiv P(x)Q(y)\left[2 + P(x)\sum_{j=1}^{s}\{b_j t_j(x, y) + a_j t_j(x^{-1}, y^{-1})\}\right],$$

$$t_j(x, y) \equiv y^{-j}(1 + x^j - 2y^j)/(1 - x^j).$$

Reducing this to a common denominator we find the form stated.

In order that $f^2(x, y)$ be separable it is therefore necessary and sufficient that

$$P(x)\sum_{j=0}^{s-1}R_j(x)y^j + 2S(x)y^s + P(x)\sum_{j=1}^{s}T_j(x)y^{s+j} \equiv L(x)M(y),$$

identically in x and y, where $L(x)$, $M(y)$ are functions of x and of y alone respectively. The trivial cases where one of $P(x)$, $L(x)$, $M(y)$ is identically zero are excluded. Hence it is necessary and sufficient that

$$P(x)R_j(x) \equiv \alpha_j L(x), \quad (j = 0, \cdots, s-1);$$
$$S(x) \equiv \beta L(x);$$
$$P(x)T_i(x) \equiv \gamma_i L(x), \qquad (i = 1, \cdots, s),$$

where the α, β, γ are constants not all zero. The first and third of these are

$$-b_{s-j}(x^{s-j}+1)A_{s-j}(x)P(x) \equiv \alpha_j L(x), \quad (j = 0, \cdots, s-1);$$
$$a_i(x^i+1)A_i(x)P(x) \equiv \gamma_i L(x), \qquad (i = 1, \cdots, s).$$

By a simple contradiction it is seen from the first that either all the b's are zero or precisely one is not zero. For, since $j<s$, the first identity gives for some j

$$(x^{s-j}+1)/(x^{s-j}-1) \equiv \delta_j L(x)P(x)/A(x),$$

where δ_j is a constant. If $\delta_j = 0$, $(j=0, \cdots, s-1)$, then $b_{s-j}=0$, $(j=0, \cdots, s-1)$. Suppose next that $\delta_j \neq 0$ when $j=j_1$, $j=j_2$, $j_1 \neq j_2$. Then the corresponding left members can differ only by a constant factor $\delta \neq 0$, and we have

$$(x^{s-j_1}+1)(x^{s-j_2}-1) \equiv \delta(x^{s-j_1}-1)(x^{s-j_2}+1).$$

Now $2s-j_1-j_2 \neq 0$, since $j_1<s$, $j_2<s$. Moreover, $s-j_1 \neq s-j_2$ since $j_1 \neq j_2$; $s-j_1 \neq 2s-j_1-j_2$; $s-j_2 \neq 2s-j_1-j_2$; $(s-j_1)(s-j_2) \neq 0$. Thus we may equate coefficients and get $\delta=1$; whence $x^{s-j_1}=x^{s-j_2}$, and we have the contradiction $j_1=j_2$.

Similarly for the second identity and the a's. There are thus four possibilities, of which one is trivial; the remaining three are as follows.

All b's and α's are zero, and all but one a are zero.

All but one of the b's are zero, and all the a's and γ's are zero.

All but one of the b's and all but one of the a's are zero.

Obviously the first two are identical on a suitable change of notation. We shall consider only the third in detail, as a similar argument shows that the conclusion reached includes the other possibilities as limiting cases.

Let then $b_{s-j}a_i \neq 0$ for a particular j and i. Write $-b_{s-j} \equiv c_{s-j}$.

Then from the two identities we have, on comparing the values which they give for $P(x)$,

$$\alpha_j a_i(x^{s-j+i} + x^{s-j} - x^i - 1) \equiv \gamma_i c_{s-j}(x^{s-j+i} - x^{s-j} + x^i - 1).$$

Now $0 \leqq j < s$, $0 < i \leqq s$. Hence $i(s-j)(s-j+i) \neq 0$, and therefore the apparent constant terms in the above are the actual constant terms, so that $\alpha_j a_i = \gamma_i c_{s-j}$. It follows thence from the last identity that $s-j=i$.

To find $L(x)$ and hence $P(x)$ apply this conclusion to $S(x)$. We find thus

$$L(x) \equiv \frac{A(x)(x^i+1)}{x^i(\beta+\gamma_i)+(\beta+\alpha_j)},$$

$$P(x) \equiv \frac{\gamma_i(x^i-1)}{a_i[x^i(\beta+\gamma_i)+(\beta+\alpha_j)]}.$$

Combining all results we have the following result.

THEOREM 2. *The unique simply separable $f(x, y)$ such that $f^2(x, y)$ is separable, is*

$$f(x, y) \equiv \frac{x^r - 1}{(a+b)x^r + (b+c)}(ay^r - cy^{-r}),$$

where r is an arbitrary constant integer $\neq 0$, and a, b, c are constants not all zero; $f^2(x, y)$ is also simply separable,

$$f^2(x, y) \equiv \frac{x^{2r}-1}{[(a+b)x^r + (b+c)]^2}(ay^r - cy^{-r})(ay^r + 2b + cy^{-r}).$$

If the further condition (3) be imposed we find immediately Theorem 1.

3. *Third Functional Equation for ϕ_k.* Let t, k, l be constants, $tkl \neq 0$. With x, y as in preceding sections, let f be a numerical function and define the numerical functions f_k, $f_{k,l}$ by

$$f_k(x, y) \equiv y^k f(x^k), \quad (f_k f_l) \equiv f_{k,l}.$$

Then by a shorter argument similar to that of §2, we find the following theorem.

THEOREM. *If $f_{k,l}(x, y) = y^t F(x^t)$, then necessarily $t = k+l$, $F = f$ and $f(x) = 1 - x^{-r}$, where r is an arbitrary constant $\neq 0$. Hence $f_k = \phi_k$ is the unique solution of the functional equation.*

CALIFORNIA INSTITUTE OF TECHNOLOGY

ABSTRACT DEFINING RELATIONS FOR THE SIMPLE GROUP OF ORDER 5616*

BY K. E. BISSHOPP

1. *Generation of* G_{5616}. The collineation group $LF(3, 3)$ of order 5616 leaves the sets of points and lines of the finite geometry $PG(3, 3)$ invariant. The points which are the sets (x_1, x_2, x_3) where x_1, x_2, x_3 are the integers modulo 3 and their corresponding polars with respect to the conic $x_1^2+2x_1x_2+x_2x_3=0$ will be denoted in the following manner.

	Point	Polar		Point	Polar
a	1 0 0	$x_1+x_2=0\theta$	g	1 0 1	$x_1=0\kappa$
b	1 1 1	$x_1+x_3=0\gamma$	h	1 2 0	$x_2+x_3=0\nu$
c	1 1 2	$x_1+x_2+x_3=0\mu$	i	1 2 1	$x_3=0\epsilon$
d	1 2 2	$x_2+2x_3=0\alpha$	j	0 1 1	$2x_1+x_2+x_3=0\delta$
e	1 1 0	$x_1+2x_2+x_3=0\iota$	k	0 0 1	$x_2=0\lambda$
f	0 1 0	$2x_1+x_3=0\zeta$	l	1 0 2	$2x_1+x_2=0\beta$
			m	0 2 1	$x_1+x_2+2x_3=0\eta$

Later we shall have occasion to refer to these polars in connection with the group of isomorphisms.

The collineation group may be generated by two operators of orders three and two respectively which satisfy the necessary conditions

$$S^3 = T^2 = (TS)^{13} = 1.$$

Two operators which belong to the group and possess these properties are

$$S = \begin{vmatrix} 2 & 2 & 1 \\ 1 & 1 & 1 \\ 2 & 1 & 0 \end{vmatrix}, \quad T = \begin{vmatrix} 2 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}.$$

For convenience in discussion we shall represent the collineation group as a permutation group on the points of the geometry. The operators S and T then become

$$S = (aib)(cfd)(ehk)(gjm),$$

$$T = (bj)(cm)(ef)(gl),$$

$$R \equiv TS = (aibmfhkedcglj).$$

* Presented to the Society, September 11, 1930.

The group generated by two operators of orders three and two respectively whose product is of the form $6k \pm 1$, where k is an integer, is perfect.* Every perfect group is simple or else it is isomorphic $(\alpha - 1)$ to some simple group of composite order. If S and T generate a subgroup of G_{5616}, its order is 432 or less. All the simple groups of orders less than 432 are of orders 60, 168, and 360. Obviously G_{5616} contains no subgroups having an operator of order 13 which is isomorphic to any of the above mentioned simple groups. Since S and T belong to G_{5616} they can generate a group of no higher order. Therefore S and T generate G_{5616}.

The group G_{5616} contains 117 operators of order two† and T is invariant under a G_{48} whose generators are

$$(TR^{-1}TR)^3 = (be)(dh)(fj)(gl),$$
$$(TR^{-3}TR^3)^2 = (bj)(hk)(eg)(fl),$$
$$(TR^{-5}TR^5) = (bc)(di)(gl)(mj).$$

The transformation of S by this G_{48} gives 48 different operators conjugate to S. The group possesses altogether 624 operators conjugate to S‡ contained in 13 sets of 48 conjugates under the G_{48} which leaves invariant. Each set contains one of the following operators:

$S_1 = (aib)(cfd)(ehk)(gjm)$, $TS_1 = (aibmfhkedcglj)$,
$S_2 = S_1^2 = (abi)(cdf)(ekh)(gmj)$,
$S_3 = R^{-1}S_1R = (afl)(bmi)(cgh)(dke)$, $TS_3 = (afdkelhcibjmg)$,
$S_4 = S_3^2 = (alf)(bim)(chg)(dek)$,
$S_5 = R^{-2}S_1R^2 = (bmf)(ced)(glk)(hji)$, $TS_5 = (bihjme)(cfd)(gk)$,
$S_6 = R^{-3}S_1R^3 = (abk)(cgd)(elj)(fhm)$, $TS_6 = (abehmgjk)(cfld)$,
$S_7 = S_6^2 = (akb)(cdg)(ejl)(fmh)$,
$S_8 = R^{-4}S_1R^4 = (adj)(cgl)(eim)(fhk)$, $TS_8 = (adjb)(cehkfimg)$,
$S_9 = S_8^2 = (ajd)(clg)(emi)(fkh)$,
$S_{10} = R^{-6}S_1R^6 = (alj)(bgi)(cmh)(dke)$, $TS_{10} = (alib)(ch)(dkef)(gj)$,

* Brahana, American Journal of Mathematics, vol. 50 (1928), p. 347.
† Miller, Quarterly Journal of Mathematics, vol. 29 (1898), p. 234.
‡ Ibid., p. 234.

$S_{11} = R^{-10}S_1R^{10} = (aec)(bmf)(hki)(glj)$, $TS_{11} = (aebgjm)(hki)(cf)$,

$S_{12} = (aih)(cke)(dfm)(glj)$,* $TS_{12} = (aih)(bgj)(cdf)(emk)$,

$S_{13} = (ajb)(cmk)(eif)(ghl)$,* $TS_{13} = (aj)(ck)(fi)(hl)$.

We will denote the sets by the subscripts on the corresponding S's. Members of the pairs (1) and (2), (3) and (4), (6) and (7), and (8) and (9) are distinct since no one of the operators of order two of the G_{48} transforms the corresponding S into its inverse. (1) and (2) are distinct from (3) and (4) since a which is omitted by G_{48} occurs in a cycle of S_1 with i while none of the letters d, h, i, k, which form a transitive constituent of G_{48}, occur in the same cycle with a in S_2. (5) and (11) are distinct since G_{48} omits a. (6) and (8) are distinct since in the product $TS\,a$ occurs in a cycle of order 8 in one case and in one of order 4 in the other. (10), (12), and (13) are distinct since the products TS are all of different orders. None of the operators $S_5, S_6, \cdots, S_{13}$ generate the group with T because the resulting groups are intransitive on 13 letters. It follows that there are two sets of operators and their inverses which generate the group with T.

THEOREM 1. *If an operator of order two and one of order three generate G_{5616}, their product is of order thirteen.*

The theorem follows since an operator of order three containing three cycles cannot generate the group with T because T can have only four cycles and the resulting group could not be transitive on 13 letters.

From §1 we obtained the following correlation of the plane:

$$(a\theta)(b\gamma)(c\mu)(d\alpha)(e\iota)(f\zeta)(g\kappa)(h\nu)(i\epsilon)(j\delta)(k\lambda)(l\beta)(m\eta) = P.$$

This is an outer isomorphism which leaves T invariant,

$$S' = (aib)(cfd)(ehk)(gjm),$$
$$T' = (bj)(cm)(ef)(gl).$$

The operator $S' = (aib)(cfd)(ehk)(gjm)$ written on points performs the permutation $\Sigma = (\alpha\nu\zeta)(\beta\epsilon\theta)(\delta\eta\kappa)(\iota\mu\lambda)$ written on lines. For a representation of S' on symbols for lines we may use Σ. The correlation P transforms points into lines,

$$P\,\Sigma\,P = (ali)(cke)(dhf)(gjm).$$

§ Operators conjugate to S_{13} and S_{13} are in the group $\{T, S_{10}\}$.

Also there exists an operator of the G_{48} which transforms $P\Sigma P$ into S^2. Such a one is $(bl)(ef)(dk)(gj)$. Since P which is an outer isomorphism transforms the operators of the first set into their inverses, every other outer isomorphism which leaves T invariant has the same property. Hence there is no outer isomorphism which leaves T invariant and at the same time transforms the first set into the second and it follows that the two sets of generators satisfy different defining relations.

2. *The Cole Group of Order* 432. Cole has investigated the groups of degree nine and found that there is a single transitive substitution group of order 432 representable on nine letters.*

The group with which we have to deal is written intransitively on thirteen letters and contains two transitive constituents one on four letters and one on nine letters. Hence it follows that the latter group is equivalent to the one described by Cole. We have a priori that the foregoing is a maximal subgroup of G_{5616}. In order to define this group abstractly it will be convenient first to determine a set of defining relations for the maximal subgroup H, of order 432.

THEOREM 2. *Two operators of orders three and two respectively which satisfy the following conditions,*

$$(1) \qquad S^3 = T^2 = (ST)^8 = 1,$$

$$(2) \qquad TS^2TS(ST)^4S^2TST = S^2TS(ST)^4S^2TS,$$

$$(3) \qquad (S^2TST)^6 = 1,$$

generate the abstract group of order 432 *which is doubly transitive on nine letters, or a group of lower order to which* G_{432} *is isomorphic.*

If S and T are of orders three and two respectively and $(ST)^4$ is identity, we get the octahedral group; if $(ST)^4$ is invariant and not identity, we get the group of order 48.

By virtue of (1), $(ST)^4$ is of order two; and as consequences of (2) and (3), it will be proved that there exists a set of nine conjugate operators arising from the transformation of $(ST)^4$ by T and powers of S. Denoting $(ST)^4$ by T_1, let us consider the following transforms:

* Bulletin of the New York Mathematical Society, vol. 2 (1892-93), pp. 254-256.

$$T_1, \qquad STST_1S^2TS^2 = T_6,$$
$$S^2T_1S = T_2, \qquad TSTST_1S^2TS^2T = T_7,$$
$$ST_1S^2 = T_3, \qquad S^2TSTST_1S^2TS^2TS = T_8,$$
$$TST_1S^2T = T_4, \qquad STSTST_1S^2TS^2TS^2 = T_9.$$
$$S^2TST_1S^2TS = T_5,$$

Obviously in order that this conjugate set contain nine and only nine operators it will be sufficient to show that T_8 and T_9 are permutable with T, that is, $TT_8T = T_8$, and $TT_9T = T_9$. The foregoing results directly from conditions (2) and (3). Writing (2) in the following form and remembering that $S^2(ST)^4S = T(ST)^4T$, we have

$$\begin{aligned}
&S^2TSTS^2TS(ST)^4S^2TSTS^2TS = (ST)^4,\\
&TS^2TSTS^2TS(ST)^4S^2TSTS^2TST = T(ST)^4T = S^2(ST)^4S,\\
&STS^2TSTS^2TS(ST)^4S^2TSTS^2TSTS^2 = (ST)^4,\\
&TSTS^2TSTS^2TS(ST)^4S^2TSTS^2TSTS^2T = T(ST)^4T,\\
&S^2TSTS^2TSTS^2TS(ST)^4S^2TSTS^2TSTS^2TS = S^2T(ST)^4TS = S(ST)^4,\\
&S^2TS^2TSTS^2TSTS^2TS(ST)^4S^2TSTS^2TSTS^2TST = TS(ST)^4S^2T.
\end{aligned} \tag{i}$$

It follows from condition (3) that $T(S^2TST)^3T = (S^2TST)^3$ and the last expression becomes

$$S^2TSTS^2TSTS^2TST(ST)^4TS^2TSTS^2TSTS^2TS = TS(ST)^4S^2T.$$

Continuing in the same manner this expression reduces to

$$STS^2TSTS^2T(ST)^4TSTS^2TSTS^2 = TSTS(ST)^4S^2TS^2T.$$

Transforming by S, we have, after replacing $T(ST)^4T$ by $S^2(ST)^4S$,

$$S^2TSTS(ST)^4S^2TS^2TS = TS^2TSTS(ST)^4S^2TS^2TST, \tag{ii}$$

which gives the desired relation between T and T_8. On transforming both members of equation (ii) above first by S and then T we have

$$TSTSTS(ST)^4S^2TS^2TS^2T = TS^2TS^2TSTS(ST)^4S^2TS^2TSTST$$
$$= TS^2TS^2TS^2TTS^2TS(ST)^4S^2TSTTSTSTST$$
(iii) $$= (ST)^3S^2(ST)^4STS^2(TS^2)^2$$
$$= STSTSTT(ST)^4S^2TS^2TS^2$$
$$= STSTS(ST)^4S^2TS^2TS^2,$$

which expresses the necessary relation between T and T_9. By the aid of (iii) we can show that T_1T_2 is of order three. Since $TT_9T = T_9$, we may write

$$STSTSTS(ST)^4S^2TS^2TS^2TS^2 = ST_9S^2 = T_8,$$
$$ST_9S^2 = (ST)^4(TS)^4(ST)^4 = T_8 = TT_8T,$$
$$(ST)^4(TS)^4(ST)^4 = (TS)^4(ST)^4(TS)^4,$$
$$T_1T_2 = (ST)^4S^2(ST)^4S = (ST)^4(TS)^4,$$
$$(T_1T_2)^3 = (ST)^4(TS)^4(ST)^4(TS)^4(ST)^4(TS)^4 = T_8^2 = 1.$$

Since T_1T_2 is of order three and T_1T_2 is transformed into T_3T_1 by S^2 the product T_1T_3 is necessarily of order three.

The group generated by T_1T_2 and T_1T_3 is abelian.* From the third of the defining relations given in the first paragraph of this section we have

$$(S^2TST)^6 = 1,$$

which implies

$$(S^2TS^2TS^2)^6 = 1,$$

and

$$S^2TS^2TS^2 = S^2(TS^2)^2 = T(ST)^5,$$
$$T(ST)^5T(ST)^5T(ST)^5T(ST)^5T(ST)^5T(ST)^5 = 1,$$
$$T(ST)^4S(ST)^4S(ST)^4S(ST)^4S(ST)^4S(ST)^4ST = 1,$$
$$(ST)^4S(ST)^4S(ST)^4S = S^2(ST)^4S^2(ST)^4S^2(ST)^4,$$
$$S^2(ST)^4S(ST)^4S(ST)^4S^2 = S(ST)^4S^2(ST)^4S^2(ST)^4S,$$
(iv) $$(ST)^4S^2(ST)^4S(ST)^4S(ST)^4S^2 = (ST)^4S(ST)^4S^2(ST)^4S^2(ST)^4S.$$

The equations (iv) express the desired relation between T_1T_2 and T_1T_3, namely:

(v) $$T_1T_2T_1T_3 = T_1T_3T_1T_2.$$

Furthermore the products T_1T_2 and T_1T_3 are different and not

* Miller, Quarterly Journal of Mathematics, vol. 33 (1901), pp. 76–79.

inverses of each other. If $T_1T_2 = T_1T_3$, $T_2 = T_3$ and this is impossible when T_1 is not invariant. Also, if $T_1T_2 = T_3T_1$, from (v) $T_3^2 = (T_1T_2)^2 = 1$, which is impossible when T_1 is not invariant since (T_1T_2) is then of order three. Hence the group $\{T_1T_2, T_1T_3\}$ is the abelian group of order nine and type (1, 1).

What has been proved so far is sufficient to state that the group generated by S and T satisfying the given conditions contains a generalized dihedral group of order 18 as an invariant subgroup since it is $\{G_9, T_1\}$ and T_1 transforms T_1T_2 and T_1T_3 into their inverses. We have shown that $(ST)^4$ is in an invariant subgroup I_{18}. Since I_{18} does not contain S or T because S is not permutable with T_1T_2 and T does not transform T_1T_3 into T_3T_1, it follows that the quotient group corresponding to I_{18} is generated by operators of orders three and two respectively, whose product is of order four, since $(ST)^4$ is in I_{18}. Hence the quotient group is the symmetric group of order 24, and the order of H is 432.

It might be observed that the group is representable on the nine congugate T's which we have discussed. Since S and T permute the T's according to the following permutations on the subscripts,

$$S = (123)(456)(789),$$

$$T = (12)(34)(67),$$

which satisfy relations (1), (2), and (3) it follows that the group is representable on nine letters and this identifies it with Cole's group. It is now clear that the structure of the entire group coincides with the description given by him, and hence it is doubly transitive.

3. *Definition of the Whole Group* G_{5616}. In §1 we discussed some of the abstract properties of the generators of the group and found that G_{5616} possesses two sets of generators which satisfy different defining relations. In order to define the whole group we shall employ the results of §2 since the largest subgroup H of order 432 forms the basis for writing all the operators of G_{5616} on co-sets. Every operator of G_{5616} occurs in one of the following co-sets,

$$\text{(i)} \qquad H, RH, R^2H, \cdots, R^{12}H.$$

Since the result of multiplication of any h_i belonging to H on the

right by any power of R is equivalent to that of multiplying another h_j on the left by some power of R it follows that integers β_i exist such that

$$\text{(ii)} \qquad \begin{gathered} R^{\alpha}\bar{S}R^{\beta_1} = h_i, \quad R^{\alpha}\bar{S}^2R^{\beta_2} = h_j, \quad R^{\alpha}TR^{\beta_3} = h_k, \\ R^{-9}SR^{9} = \bar{S}, \text{ and } H = \{\bar{S}, T\}, \end{gathered}$$

are true for all values of α from 1 to 12 inclusive.

On the basis of

$$\text{(iii)} \qquad R^{2}\bar{S}R^{10} = h_1, \quad R^{3}\bar{S}R^{6} = h_2, \quad R^{4}\bar{S}R^{8} = h_3,$$

where h_1, h_2, h_3 are operators of H, we shall show that the β_i's of equations (ii) are uniquely determined and that h_i, h_j, h_k are expressible in terms of the generators of H. Writing $R^{\alpha}\bar{S}^2R^{\beta}$ in the form $R^{\alpha-1}R^{-9}TSS^2TSR^9R^{\beta-1}$, we see that it is equal to $R^{\alpha-1}\bar{S}R^{\beta-1}$. We can treat $R^{\alpha}TR^{\beta}$ in the same manner. Since $R^{\alpha}TR^{\beta} = R^{\alpha}TTSR^{\beta-1} = R^{\alpha}SR^{\beta-1} = R^{\alpha+9}R^{-9}SR^9R^{\beta-10}$, we have $R^{\alpha}TR^{\beta} = R^{\alpha+9}\bar{S}R^{\beta-10}$. These two statements are sufficient to express the equations (ii) in terms of h_1, h_2, h_3, or, what is the same, in terms of the generators of H. We proceed as follows:

$$h_1h_2 = R^{2}\bar{S}R^{10}R^{3}\bar{S}R^{6} = R^{2}\bar{S}^2R^{6} = R\bar{S}R^{5},$$

where h_1, h_2, h_3 are given by (iii). We have also

$$\begin{aligned} h_2^{-1}h_3 &= R^{7}\bar{S}^2R^{10}R^{4}\bar{S}R^{8} = R^{6}\bar{S}R^{9}R^{4}\bar{S}R^{8} = R^{6}\bar{S}^2R^{8} = R^{5}\bar{S}R^{7}, \\ h_2^{-1}h_3^{2} &= R^{6}\bar{S}^2R^{8}R^{5}\bar{S}^2R^{9} = R^{6}\bar{S}R^{9}, \\ (h_1h_2)^{-1} &= R^{8}\bar{S}^2R^{12} = R^{7}\bar{S}R^{11}, \\ (h_1h_2)^{-1}\bar{S} &= R^{8}\bar{S}^2R^{12}R\bar{S}^2R = R^{8}\bar{S}R, \\ T &= SS^2T = R^{9}R^{-9}SR^9R^3 = R^{9}\bar{S}R^{3}, \\ [(h_1h_2)^{-1}\bar{S}]^{-1}T &= R^{11}\bar{S}R^{4}R^{9}\bar{S}R^{3} = R^{11}\bar{S}^2R^{3} = R^{10}\bar{S}R^{2}, \\ [(h_1h_2)^{-1}\bar{S}]^{-1} &= R^{12}\bar{S}^2R^{5} = R^{11}\bar{S}R^{4}, \quad \bar{S}^2 = (R\bar{S}^2R)^{-1} = R^{12}\bar{S}R^{12}. \end{aligned}$$

Since $R^{\alpha}\bar{S}R^{\beta} = R^{\alpha+1}\bar{S}^2R^{\beta+1}$, we have for the expressions $R^{\alpha}\bar{S}^iR^{\beta i}$ where $\alpha = (1, 2, 3, \cdots, 12)$:

$R\bar{S}^2R$	$R^{4}\bar{S}^2R^{7}$	$R^{7}\bar{S}^2R^{10}$	$R^{10}\bar{S}^2R^{4}$
$R^{2}\bar{S}^2R^{6}$	$R^{5}\bar{S}^2R^{9}$	$R^{8}\bar{S}^2R^{12}$	$R^{11}\bar{S}^2R^{3}$
$R^{3}\bar{S}^2R^{11}$	$R^{6}\bar{S}^2R^{8}$	$R^{9}\bar{S}^2R^{2}$	$R^{12}\bar{S}^2R^{5}$

Also for $R^{\alpha+9}\bar{S}R^{\beta-10}$, we have $R^{\alpha}TR^{\beta}$, where α ranges over the same values as above:

$$RTR^{12} = R^{10}\bar{S}R^{2},\ R^{3}TR^{2} = R\bar{S}R^{5},\quad R^{9}TR^{4} = R^{5}\bar{S}R^{7},$$

$$R^{2}TR = R^{11}\bar{S}R^{4},\ R^{6}TR^{7} = R^{2}\bar{S}R^{10},\ R^{10}TR^{6} = R^{6}\bar{S}R^{9},$$

$$R^{3}TR^{9} = R^{12}\bar{S}R^{12} = \bar{S}^{2},\quad R^{7}TR^{3} = R^{3}\bar{S}R^{6},\ R^{11}TR^{8} = R^{7}\bar{S}R^{11},$$

$$R^{4}TR^{10} = \bar{S},\quad R^{8}TR^{5} = R^{4}\bar{S}R^{8},\ R^{12}TR^{10} = R^{8}\bar{S}R.$$

Hence all of the equations (ii) have solutions in terms of h_1, h_2, h_3, and the β_i's are uniquely determined. In a similar manner when $\bar{S}=R^{-1}SR$ which gives another set of conditions we can also show that the equations are reducible to expressions in terms of the generators of H and that the β_i's are uniquely determined.

Having shown that we can establish sufficient conditions which enable us to specify the order of G, we may now state the following theorem.

THEOREM 3. *Two operators T and S of orders two and three with product of order thirteen generate G_{5616} if and only if they satisfy the further relations*:

$$\bar{S}^{3} = T^{2} = (\bar{S}T)^{8} = 1, \tag{1}$$

$$T\bar{S}^{2}T\bar{S}(\bar{S}T)^{4}\bar{S}^{2}T\bar{S}T = \bar{S}^{2}T\bar{S}(\bar{S}T)^{4}\bar{S}^{2}T\bar{S}, \tag{2}$$

$$(\bar{S}^{2}T\bar{S}T)^{6} = 1, \tag{3}$$

where $\bar{S}=R^{-9}SR$ or $R^{-1}SR$ and $R\equiv TS$, and either (4), (5), (6), *or* (4′), (5′), (6′),

$$R^{2}\bar{S}R^{10} = T\bar{S}^{2}T\bar{S}T, \tag{4}$$

$$R^{3}\bar{S}R^{8} = T\bar{S}(\bar{S}T)^{4}\bar{S}(\bar{S}T)^{4}\bar{S}^{2}T\bar{S}^{2}, \tag{5}$$

$$R^{4}\bar{S}R^{3} = T\bar{S}^{2}T\bar{S}T\bar{S}(\bar{S}T)^{4}\bar{S}^{2}T\bar{S}^{2}T\bar{S}, \tag{6}$$

where $\bar{S}=R^{-9}SR$;

$$R^{2}\bar{S}R^{9} = T\bar{S}^{2}T\bar{S}T, \tag{4′}$$

$$R^{5}\bar{S}R^{4} = T\bar{S}^{2}T\bar{S}^{2}T\bar{S}(\bar{S}T)^{4}\bar{S}^{2}T, \tag{5′}$$

$$R^{7}\bar{S}R^{5} = \bar{S}(\bar{S}T)^{4}\bar{S}(\bar{S}T)^{3}\bar{S}, \tag{6′}$$

where $\bar{S}=R^{-1}SR$.

If two pairs of operators ρ and τ and Σ and τ which generate the group are so selected that one of them ρ and τ satisfy a particular one of the sets of conditions imposed on S and T and if ρ happens to be the inverse of Σ then Σ and τ satisfy identically the same relations as ρ and τ. This follows from the fact that there is an outer isomorphism which transforms ρ into its inverse and leaves T invariant.

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ONE-PARAMETER LINEAR FUNCTIONAL GROUPS IN SEVERAL FUNCTIONS OF TWO VARIABLES*

BY A. D. MICHAL

Let

$$(1) \qquad {}_i^jH(x, y), \qquad (i, j = 1, 2, \cdots, n),$$

be a given set of n^2 real continuous functions of the two real variables x and y defined over the triangular region $T: a \leqq x \leqq y \leqq b$. Consider the system of n *integro-differential equations*†

$$(2) \qquad \frac{\partial {}_iz(x, y; \tau)}{\partial \tau} = {}_i^jH \star {}_jz(x, y; \tau), \quad (i = 1, 2, \cdots, n),$$

in the n unknown functions ${}_1z(x, y; \tau)$, ${}_2z(x, y; \tau)$, $\cdots$, ${}_nz(x, y; \tau)$. The symbol $\star$ in (2) stands for *Volterra's operation composition of the first kind*

$$(3) \qquad {}_i^jH \star {}_jz(x, y; \tau) = \int_x^y {}_i^jH(x, s)\, {}_jz(s, y; \tau)ds.$$

We assume that the reader is conversant with the theory of permutable functions and functions of composition, a subject initiated by the illustrious Vito Volterra.‡ Griffith C. Evans§

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* Presented to the Society, December 30, 1930.

† Throughout the whole paper we shall adhere to the convention of letting a repetition of an index in a term once as a subscript and once as a superscript denote summation with respect to that index over the values $1, 2, \cdots, n$.

‡ See, for example, Volterra et Pérès, *Leçons sur la Composition et les Fonctions Permutables.*

§ See his Cambridge Colloquium Lectures, *Functionals and their Applications.*

has given some very noteworthy contributions to this subject.

Define a set of n^2 functions ${}_i^jK(x, y; \tau)$ in terms of the given set of functions ${}_i^jH(x, y)$ by

$$(4)\qquad {}_i^jK(x, y; \tau) = {}_i^jH(x, y)\tau + {}_i^kH\star_k^jH(x, y)\frac{\tau^2}{2!} + {}_i^kH\star_k^lH\star_l^jH(x, y)\frac{\tau^3}{3!} + \cdots.$$

For $n=1$, this set reduces to the well known Volterra transcendental.

THEOREM 1. *Under the above hypotheses on the given functions ${}_i^jH(x, y)$, the system of integro-differential equations* (2) *possesses one and only one solution ${}_1z(x, y; \tau), \cdots, {}_nz(x, y; \tau)$ analytic in τ (in fact entire in τ), continuous in x and y in the region T and such that it takes on the initial conditions*

$$(5)\qquad {}_iz(x, y; 0) = {}_iz(x, y),$$

(${}_iz(x, y)$ continuous in T) for $\tau=0$. This unique solution can be written in the form

$$(6)\qquad {}_iz(x, y; \tau) = {}_iz(x, y) + {}_i^jK(\cdots; \tau)\star_jz(x, y),$$

where the n^2 functions ${}_i^jK$ are the generalized Volterra transcendentals (4).

The proof of this theorem is obtained by throwing the integro-differential system (2) into the form of a system of non-linear integral equations

$$(7)\qquad {}_iz(x, y; \tau) = {}_iz(x, y) + \int_0^\tau {}_i^jH\star_jz(x, y; \sigma)d\sigma,$$

and then solving this system by the method of successive approximations. We are thus led to consider the functions ${}_iz(x, y; \tau)$ given by (6). By calculation one sees that this set of functions is a formal solution of the integro-differential system and takes on the given initial conditions. In fact the transcendentals ${}_i^jK(x, y; \tau)$ satisfy the integro-differential equations

$$(8)\qquad \frac{\partial {}_i^jK(x, y; \tau)}{\partial\tau} = {}_i^jH(x, y) + {}_i^kH\star_k^jK(x, y; \tau),$$

and hence the ${}_iz(x, y; \tau)$ of (6) yield a formal solution of the system (2).

It follows directly from the definition (4) that the generalized Volterra transcendentals are *entire* functions of τ and continuous in x and y throughout the region T. Hence the set of functions (6) form an actual solution of (2) of the required sort.

If ${}_iw(x, y; \tau)$ is another solution of (2) of the required sort, it follows from (7) that it is possible to find a positive number $A < 1$ such that

$$\max \left| {}_iz(x, y; \tau) - {}_iw(x, y; \tau) \right| \leqq A \max \left| {}_iz(x, y; \tau) - {}_iw(x, y; \tau) \right|,$$

for sufficiently small values of τ. But ${}_iz$ and ${}_iw$ are analytic functions of τ. Hence

$${}_iz(x, y; \tau) \equiv {}_iw(x, y; \tau).$$

This shows that the solution of (2) of the required sort is unique, which completes the proof of our theorem.

Thus the integro-differential system (2) generates a continuous one-parameter family of functional transformations (6). Moreover, one can verify that the generalized Volterra transcendentals ${}_i^jK$ possess the following integral addition theorem:

$$(9)\quad {}_i^jK(x, y; \tau_1 + \tau_2) = {}_i^jK(x, y; \tau_1) + {}_i^jK(x, y; \tau_2) + {}_k^jK(\cdots; \tau_2) \star_i^k K(x, y; \tau_1),$$

and that this integral addition theorem characterizes the n^2 transcendentals ${}_i^jK(x, y; \tau)$. We have thus arrived at the following fundamental theorem.

THEOREM 2. *The integro-differential equations* (2) (*infinitesimal transformations*) *generate the one-parameter continuous group of linear functional transformations* (6) *in the functions* ${}_1z(x, y)$, ${}_2z(x, y), \cdots, {}_nz(x, y)$. *Moreover the identity transformation is obtained by putting the parameter* τ *equal to zero.*

Consider the particular case in which

$$(10)\quad \begin{cases} {}_i^jH(x, y) = 1, & \text{if } j = n - i + 1, \\ \qquad = 0, & \text{otherwise.} \end{cases}$$

The integro-differential system (2) for this case becomes

$$\frac{\partial_i z(x, y; \tau)}{\partial \tau} = 1_{\star_{n-i+1}} z(x, y; \tau). \tag{11}$$

By calculation we obtain

$$\begin{cases} {}_i^j K(x, y; \tau) = \sum_{n=1}^{\infty} \dfrac{\tau^{2n}(y-x)^{2n-1}}{(2n)!(2n-1)!}, & \text{if } j = i, \\ \qquad = \sum_{n=1}^{\infty} \dfrac{\tau^{2n-1}(y-x)^{2n-2}}{(2n-1)!(2n-2)!}, & \text{if } j = n-i+1, \\ \qquad = 0, \text{ otherwise.} \end{cases} \tag{12}$$

These expressions can be written in terms of the Bessel function $J_1(z)$ of order one.* Putting $u = y - x$, we obtain

$$\begin{cases} {}_i^j K(x, y; \tau) = \dfrac{\tau^{1/2} u^{-1/2}}{2} \{ I_1[2(\tau u)^{1/2}] + J_1[-2(\tau u)^{1/2}] \}, & \text{if } j = i, \\ \qquad = \dfrac{\tau^{1/2} u^{-1/2}}{2} \{ I_1[2(\tau u)^{1/2}] - J_1[-2(\tau u)^{1/2}] \}, & \text{if } j = n-i+1, \\ \qquad = 0, \text{ otherwise.} \end{cases} \tag{13}$$

In these formulas,

$$I_1(z) = -iJ_1(iz), \quad (i = (-1)^{1/2}).$$

For $j = i$ the integral addition theorems (9) for the special case (13) yield the addition theorem

$$\begin{cases} (\tau_1 + \tau_2)^{1/2} u^{-1/2} A[(\tau_1 + \tau_2)u] \\ \quad = \tau_1^{1/2} u^{-1/2} A[\tau_1 u] + \tau_2^{1/2} u^{-1/2} A[\tau_2 u] \\ \qquad + (\tau_1 \tau_2)^{1/2} \displaystyle\int_0^u [(u-v)v]^{-1/2} \\ \qquad \cdot \{ J_1[-2(\tau_2(u-v))^{1/2}] J_1[-2(\tau_1 v)^{1/2}] \\ \qquad + I_1[2(\tau_2(u-v))^{1/2}] I_1[2(\tau_1 v)^{1/2}] \} dv, \end{cases} \tag{14}$$

where

$$A[\tau u] = J_1[-2(\tau u)^{1/2}] + I_1[2(\tau u)^{1/2}].$$

* A. D. Michal, Annals of Mathematics, vol. 26 (1925), for the corresponding case of the Volterra transcendental.

By an application of a known theorem* one can verify that $I_1(z)$ possesses the following integral addition theorem:

$$(15)\quad \begin{cases} (\tau_1+\tau_2)^{1/2}u^{-1/2}I_1[2((\tau_1+\tau_2)u)^{1/2}] \\ = \tau_1^{1/2}u^{-1/2}I_1[2(\tau_1u)^{1/2}] + \tau_2^{1/2}u^{-1/2}I_1[2(\tau_2u)^{1/2}] \\ + (\tau_1\tau_2)^{1/2}\int_0^u [(u-v)v]^{-1/2}I_1[2(\tau_2(u-v))^{1/2}]I_1[2(\tau_1v)^{1/2}]dv. \end{cases}$$

Moreover

$$J_1(z) = iI_1(-iz), \qquad (i=(-1)^{1/2}).$$

Hence by calculation we see that $J_1[-2(\tau u)^{1/2}]$ has precisely the same integral addition theorem as that of $I_1[2(\tau u)^{1/2}]$.

The remaining relations (9) for the particular system (10) do not yield any essentially new integral addition theorems for the Bessel functions.

THEOREM 3. *The group properties of the continuous one-parameter family of functional transformations that is generated by the integro-differential system* (11) *are translated by the integral addition theorem*

$$(\tau_1+\tau_2)^{1/2}u^{-1/2}J_1[-2((\tau_1+\tau_2)u)^{1/2}]$$
$$= \tau_1^{1/2}u^{-1/2}J_1[-2(\tau_1u)^{1/2}] + \tau_2^{1/2}u^{-1/2}J_1[-2(\tau_2u)^{1/2}]$$
$$+ (\tau_1\tau_2)^{1/2}\int_0^u [(u-v)v]^{-1/2}J_1[-2(\tau_2(u-v))^{1/2}]J_1[-2(\tau_1v)^{1/2}]dv$$

for the Bessel function $J_1(z)$ of order one.

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* H. P. Thielman, Annals of Mathematics, vol. 31 (1930), p. 193.

THE PROBABLE ERROR OF CERTAIN FUNCTIONS OF THE ERRORS MADE IN MEASUREMENTS*

BY W. D. BATEN

1. *Introduction.* This paper presents with proof and applications several theorems pertaining to the probable error of certain functions of the errors made in measurements. If the measurements or observations are multiplied by a constant b, the probable value of functions of the error of bm will differ from the probable value of these same functions of the error of m the measurement. This article shows that there are values of b such that the probable value of certain functions of the error of bm is less than the probable value of these functions of the error of m.

The probable value of certain functions of the mean are also treated. General frequency laws for the errors are used in the first theorems; these include the discrete, the continuous and a combination of the discrete and the continuous cases. Here Stieltjes integrals are used in the proofs. Special cases are mentioned.

The following theorems are proven by use of general frequency laws which come under the continuous case. The Gaussian law is treated as special cases to the theorems.

2. *Concerning the Square of the Error of the Measurement.*

THEOREM 1. *Let x be the error of the measurement m, d the expected value of x, and c the expected value of x^2. Then under any law of error, whose second moment with respect to the true value a exists, there exist values of the constant b such that the probable value of the square of the error of bm is less than the probable value of the square of the error of the measurement, provided*

$$ad - c \neq 0, \qquad a^2 - 2ad + c \neq 0.$$

Under these conditions b lies between

$$1 \quad \text{and} \quad 1 - \frac{2(c - ad)}{a^2 - 2ad + c}.$$

* Presented to the Society, April 19, 1930.

PROOF. Let $F(x)$ be the probability that x lies in the interval $(-\infty, x)$ where x is included. Then the probability that x lies in $(-\infty, x^*)$, where x^* means that x has been deleted, is $F(x-0)$. The probability that x lies at x is $F(x+0)-F(x-0)$, the probability that x lies in the segment (x_1^*, x_2^*) is

$$F(x_2 - 0) - F(x_1 + 0) = \int_{x_1+0}^{x_2-0} dF(x).$$

Finally the probability that x lies in the closed interval (x_1, x_2) is

$$F(x_2) - F(x_1 - 0) = \int_{x_1-0}^{x_2} dF(x).$$

We have also $F(-\infty)=0$ and $F(+\infty)=1$. If $f(x)$ is the probability function and is continuous, then

$$F(x) = \int_{-\infty}^{x} f(x)dx. \tag{1}$$

The error of the measurement m from the true value a is $x=a-m$. The error of bm is

$$x' = a - bm = a - b(a - x) = a(1 - b) + bx.$$

The probable value of $(x')^2$ is

$$\left\{\begin{aligned} P(x'^2) &= \int_{-\infty}^{\infty} \left\{a^2(1 - b)^2 + 2a(1 - b)bx + b^2x^2\right\} dF(x) \\ &= a^2(1 - b)^2 + 2abd(1 - b) + b^2c, \end{aligned}\right. \tag{2}$$

which is less than the probable value of the square of the error of m,

$$P(x^2) = \int_{-\infty}^{\infty} x^2 dF(x) = c,$$

if

$$a^2(1 - b)^2 + 2abd(1 - b) + b^2c < c.$$

From this inequality it is seen that b lies between

$$1 \quad \text{and} \quad 1 - \frac{2(c - ad)}{a^2 - 2ad + c}.$$

If the law of error is a Pearson type III,

$$y = k\left(1 + \frac{x}{q}\right)^{hq} e^{-hx},$$

then b lies between

$$1 \text{ and } 1 - \frac{2ke^{hq}(hq - ah + 2)\Gamma(hq+1)}{a^2h^2(hq)^{hq} - 2akhe^{hq}\Gamma(hq+1) + ke^{hq}(qh+2)\Gamma(hq+1)},$$

provided the denominator is not zero.

COROLLARY. *Under any law of error, whose second moment with respect to the true value a exists and whose first moment with respect to the true value is zero, there exist values of the constant b such that the probable value of the square of the error of bm is less than the probable value of the square of the error of m. These values satisfy the inequalities*

$$1 - \frac{2c}{a^2 + c} < b < 1.$$

The laws of error considered in the corollary include those which are symmetrical with respect to the true value and also others which are skew.

The preceding theorem does not apply to the law of error

$$f(x) = \frac{h}{\pi(1 + h^2x^2)},$$

for the second moment does not exist.

If the law of error for the error of m is

$$f(x) = \frac{2h}{\pi(1 + h^2x^2)^2}, \tag{3}$$

then b satisfies the inequalities

$$1 - \frac{2}{a^2h^2 + 1} < b < 1.$$

If the error law is a Pearson type X†,

$$f(x) = \frac{h}{2}e^{-h|x|}, \tag{4}$$

then b satisfies the inequalities

† Elderton, *Frequency Curves and Correlation.*

$$1 - \frac{4}{a^2h^2 + 2} < b < 1.$$

If the error law is the Gaussian law,

$$\frac{h}{\sqrt{\pi}} e^{-h^2x^2},$$

then the corollary to Theorem 1 becomes in this special case identical to a theorem given by Dodd.*

If the error law for the errors of the measurement m is $f(x) = 1/(2k)$ in $(-k, k)$ and zero elsewhere, then

$$1 - \frac{2k^2}{3a^2 + k^2} < b < 1.$$

This last law is used for the law of errors which are made in using a table of logarithms, for here the error is constant in a certain interval and zero elsewhere.

3. *Concerning the Error of the Mean.* If M is the mean and B is a constant it is desirable to know the nature of the probable value of certain functions of the error of BM and also the probable value of these same functions of the error of M. The following theorems will be useful in this respect.

THEOREM 2. *If the error law for the individual variable is of limited variation and symmetrical with respect to the true value, then the error law for the mean is also symmetrical with respect to the true value.*†

PROOF. Let $f(x)dx$ be the probability that the error lies in the interval $(x, x+dx)$. The probability that the sum lies in the interval $(u, u+du)$ is, according to a theorem given by Dodd,‡

$$p_n(u) = \frac{1}{\pi} \int_0^\infty g(t) \cos(ut)dt,$$

* Dodd, *The error-risk of certain functions of the measurements*, Monatshefte für Mathematik und Physik, vol. 24 (1913), pp. 268–276; Dodd, *The probability of the arithmetic mean compared with that of certain other functions of the measurements*, Annals of Mathematics, (2), vol. 14 (1912–13), pp. 186–198.

† Theorem 2 can be proved for the discrete case.

‡ Dodd, *The frequency law of a function of variables with given frequency laws*, Annals of Mathematics, (2), vol. 27 (1925–26), pp. 12–20.

where the parameter u does not enter the function $g(t)$. The function $p_n(u)$ is an even function and hence is symmetrical with respect to the origin which has been chosen as the true value. The law for the mean is obtained by replacing $p_n(u)$ by $p_n(nu)n$, and this is symmetrical to the true value.

In the theory of sampling from a parent population which is known, the standard error of the mean is equal to the standard error of the parent population divided by the square root of the number of the variates in the sample, provided the parent contains an infinite number of variates.* If the true value of the quantity to be measured is a and is considered to be the mean of the parent population the standard error of the mean can in general be found for the mean from that of the error law for the individual measurements. This idea is treated also in the theory of expected values leading up to the Tchebycheff inequalities.†

When the error laws for the individual errors of the measurements are the same, then the expected value of the mean of the errors is the same as the expected value of an individual error. This can be treated also by sampling theory.

THEOREM 3. *Let x be the error of the measurement m, d the expected value of x and c the expected value of x^2. Then under any law of error whose second moment with respect to the true value a exists, there exist values of the constant B such that the probable value of the square of the error of BM is less than the probable value of the square of the error of M, provided*

$$adn - c \neq 0, \quad a^2n - 2adn + c \neq 0.$$

Under these conditions B lies between

$$1 \quad \text{and} \quad 1 - \frac{2(c - adn)}{a^2n - 2adn + c}.$$

The proof of this theorem is similar to that of Theorem 1.

COROLLARY. *Under any law of error, whose second moment exists and whose first moment with respect to the true value a is zero, there exist values of the constant B such that the probable value*

* C. H. Richardson, *The Statistics of Sampling*, a dissertation for the doctorate at The University of Michigan.

† Fisher, *The Mathematical Theory of Probability*, pp. 104–109.

of the square of the error of BM is less than the probable value of the square of the error of M; these values satisfy the inequalities

$$1 - \frac{2c}{na^2 + c} < B < 1,$$

where M is the mean of n measurements and c is the expected value of the square of the error.

The proof of this corollary is similar to that of the Corollary to Theorem 1, since Theorem 2 shows that the error law for the mean is such that its first moment with respect to the true value is also zero.

If the law for the individual errors is (3), then the law for the mean of the errors is*

$$\frac{1}{\pi(hn)^n} \sum_{r=1}^{n+1} \binom{n}{r-1} \frac{\Gamma(r) \cos(r \cdot \tan^{-1} u)}{(hn)^{-n-1}(n^2 + n^2h^2u^2)^{r/2}}.$$

In this case the constant B satisfies the following inequalities:†

$$1 - \frac{2}{na^2h^2 + 1} < B < 1,$$

where n measurements are made.

If the error law for the individual measurement is (4), then the law for the mean is‡

$$nhe^{-nh|u|} \sum_{r=0}^{n-1} \frac{n^{(-r)}hn\,|\,u\,|^{n-1-r}}{2^r(n-1-r)!(r)!}.$$

In this case

$$1 - \frac{4}{na^2h^2 + 2} < B < 1.$$

If the error law for the errors of the measurement m is $f(x) = 1/(2k)$, then

* Baten, *Theorems concerning probability*, a dissertation for the doctorate at the University of Michigan, 1929. Karl Mayr, *Wahrscheinlichkeitsfunktionen und ihre Anwendungen*, Monatshefte für Mathematik und Physik, vol. 30 (1920), pp. 17–44.

† These inequalities for B can be obtained most readily from the three inequalities for b at the end of §2 by replacing h^2 by nh^2; and the validity of this process follows from the second paragraph above Theorem 3.

‡ See Baten, loc. cit.

$$1 - \frac{2k^2}{3na^2 + k^2} < B < 1.$$

If the error law is

$$f(x) = \frac{2h}{\pi(e^{hx} + e^{-hx})},$$

then

$$1 - \frac{2\pi^2}{4a^2h^2 + \pi^2} < b < 1,$$

and

$$1 - \frac{2\pi^2}{4na^2h^2 + \pi^2} < B < 1.$$

If the error function is

$$f(x) = \frac{ph}{2\Gamma(1/p)} e^{-h^p|x|^p},$$

we shall have

$$1 - \frac{2\Gamma(3/p)}{\Gamma(1/p)na^2h^2 + \Gamma(3/p)} < b < 1,$$

and

$$1 - \frac{2\Gamma(3/p)}{\Gamma(1/p)na^2h^2 + \Gamma(3/p)} < B < 1.$$

Other error laws may be used.

4. *Concerning the Absolute Error of the Measurement.*

THEOREM 4. *Under any law of error which is symmetrical with respect to the true value a, whose first moment with respect to the true value exists and which has only one maximum which is at the true value, there exist values of the constant b such that the probable value of the absolute error of bm is less than the probable value of the absolute error of the measurement m. A sufficient condition for such a value is*

$$1 - \frac{r}{ka^2 + r} < b < 1,$$

where r is one-half of the mean deviation and k is the maximum value, and a is assumed to be positive.

PROOF. The error $|x'| = |a - bm| = |a(1-b) + bx|$, where x is the error of the measurement m. The probable value of $|x'|$ is

$$P(|x'|) = \int_{-\infty}^{\infty} |a(1-b)+bx| f(x)dx,$$

where $f(x)$ is the error law. The quantity $\{a(1-b)+bx\}$ is negative when

$$x < -\frac{a(1-b)}{b} = -R.$$

Hence

$$\begin{aligned}
P(|x'|) &= -\int_{-\infty}^{-R} \{a(1-b)+bx\} f(x)dx \\
&\quad + \int_{-R}^{\infty} \{a(1-b)+bx\} f(x)dx \\
&= a(1-b)\left\{-\int_{-\infty}^{-R} f(x)dx + \int_{R}^{\infty} f(x)dx\right\} \\
&\quad + a(1-b)\int_{-R}^{R} f(x)dx + b\left\{-\int_{-\infty}^{-R} xf(x)dx \right. \\
&\quad \left. + \int_{R}^{\infty} xf(x)dx\right\} + b\int_{-R}^{R} xf(x)dx \\
&= a(1-b)\int_{-R}^{R} f(x)dx + 2b\int_{R}^{\infty} xf(x)dx \\
&< a(1-b)\,2ka(1-b)/b + 2b\int_{0}^{\infty} xf(x)dx,
\end{aligned}$$

which is less than

$$P(|x|) = \int_{-\infty}^{\infty} |x| f(x)dx,$$

if

$$2a^2k(1-b)^2/b + 2br < 2\int_{0}^{\infty} xf(x)dx,$$

or if

$$2a^2k(1-b)^2 + 2b^2r - 2br < 0,$$

or if

$$1 - \frac{r}{a^2k+r} < b < 1.$$

If the error law is the Gaussian law then the above result is identical to that obtained by Dodd who used altogether the Gaussian law.

THEOREM 5. *Under the law* $f(x) = he^{-h|x|}/2$, *there exist values of the constant* b *such that the probable value of the absolute error of* bm *is less than the probable value of the absolute error of the measurement* m. *A sufficient condition for such a value is*

$$1 - \frac{1 - ah}{a^2h^2 - ah + 1} < b < 1,$$

where a *is positive.*

PROOF. By a similar process,

$$P(|x'|) = ha(1 - b)\int_0^R e^{-hx}dx + hb\int_R^\infty xe^{-hx}dx$$

$$= a(1 - b)(1 - e^{-hR}) + bRe^{-hR} + be^{-hR}/h.$$

Since

$$e^{hR} = (1 + hR + h^2R^2/2 + \cdots),$$

we have $e^{hR} > hR$ and $e^{-hR} < 1/(hR)$. We have also

$$1 - e^{-hR} = \left(1 - 1 + hR - \frac{h^2R^2}{2!} + \frac{h^3R^3}{3!} - \cdots\right)$$

$$= hR - \left(\frac{h^2R^2}{2!} - \frac{h^3R^3}{3!} + \cdots\right).$$

The last parenthesis is positive, therefore

$$1 - e^{-hR} < hR = a(1 - b)h/b.$$

Therefore

$$P(|x'|) \leqq ha^2(1 - b)^2/b + hR + b/h,$$

which is less than $P(|x|) = 1/h$ if

$$h^2a^2 - 2a^2h^2b + a^2h^2b^2 + hba(1 - b) + b^2 < b,$$

or

$$1 - \frac{1 - ah}{a^2h^2 - ah + 1} < b < 1.$$

The denominator is not zero unless ah, a, or h is imaginary. These cases are excluded here. If $1 > ah$ the fraction in the first inequality is positive. For other cases the inequalities have no meaning. Theorem 4 gives a lower limit for b regardless of the true value of a.

If the error law is

$$f(x) = \frac{2h}{\pi(1 + h^2x^2)^2},$$

we have

$$1 - \frac{1}{2a^2h^2 + 1} < b < 1.$$

Under the law

$$f(x) = \left(\frac{h}{4} + \frac{h^2|x|}{4}\right) e^{-h|x|},$$

we have

$$1 - \frac{3}{a^2h^2 + 3} < b < 1.$$

Under a type of Bayes function

$$f(x) = \frac{3h(1 - h^2x^2)}{4},$$

we have

$$1 - \frac{1}{4a^2h^2 + 1} < b < 1.$$

If

$$f(x) = \frac{2h}{\pi(e^{hx} + e^{-hx})},$$

we have

$$1 - \frac{\sum_0^\infty (-1)^n/(2n+1)^2}{ha^2 + \sum_0^\infty (-1)^n/(2n+1)^2} < b < 1.$$

Under the error law

$$f(x) = \frac{hp}{2\Gamma(1/p)} e^{-h^p|x|^p},$$

we have

$$1 - \frac{\Gamma(2/p)}{ph^2a^2 + \Gamma(2/p)} < b < 1.$$

Under the law $f(x) = 1/(2k)$, where k is a constant, we have

$$1 - \frac{k^2}{2a^2 + k^2} < b < 1.$$

5. *Concerning the Absolute Value of the Error of the Mean.* If the Gaussian law is the error law for the individual errors, then the law for the mean of n is also another Gaussian law. According to Theorem 4, we may write

$$1 - \frac{1}{2h^2a^2\sqrt{n}+1} < B < 1, \tag{5}$$

where B is a constant which is multiplied by the mean M. This indicates that the mean deviation of the mean is larger than the mean deviation of B times the mean, provided the law of error is the Gaussian law. This indicates that there is a better value than the mean for the true value, if the law is the Gaussian law.

The method of random sampling from a known parent population may be employed to investigate the nature of B for other laws of error. Let the parent population consist of the variates $x_1, x_2, x_3, \cdots, x_s$. Let samples of r variates be taken from this parent. It has been shown† that the mean of all possible sample means is equal to the mean of the parent population, if repetitions of samples are not allowed.

Let the first sample mean of r variates be

$$Z_1 = \frac{x_1 + x_2 + x_3 + \cdots + x_r}{r},$$

then the deviation of the first sample mean from the mean of the sample means is, in absolute value,

$$\begin{aligned} |\bar{Z}_1| &= \left| \frac{x_1 + x_2 + x_3 + \cdots + x_r}{r} - M_x \right| \\ &= \left| \frac{(x_1 - M_x) + (x_2 - M_x) + (x_3 - M_x) + \cdots + (x_r - M_x)}{r} \right| \\ &= \left| \frac{\bar{x}_1 + \bar{x}_2 + \bar{x}_3 + \cdots + \bar{x}_r}{r} \right| \\ &\leqq \frac{|\bar{x}_1| + |\bar{x}_2| + |\bar{x}_3| + \cdots + |\bar{x}_r}{r}, \qquad \bar{x}_i = x_i - M_i, \end{aligned}$$

where the mean of the parent population is M_x.

† An editorial, Annals of Mathematical Statistics, vol. 1 (1930), pp. 101–121. See also the first footnote on page 61.

The mean deviation of the sample means is found by finding the average of all possible $|\overline{Z}|$'s and this is clearly less than or equal to the mean deviation of the parent population. The mean deviation of the sample means is

$$\frac{\sum_{i=1}^{{}_sC_r} |\overline{Z}_i|}{{}_sC_r},$$

while the mean deviation of B times the sample means is

$$\frac{\sum_{i=1}^{{}_sC_r} B|\overline{Z}_i|}{{}_sC_r} = \frac{B\sum_{i=1}^{{}_sC_r} |\overline{Z}_i|}{{}_sC_r},$$

where the summations are divided by ${}_sC_r$ since there are that many possible samples that can be taken from the s variates. The mean deviation of B times the sample means is less than the mean deviation of the sample means if B is less than unity. If s goes to infinity the above is also true.

The above inequalities for the Gaussian law give more information than the above sampling method. B is not only less than 1 but it lies in a certain interval with the end points deleted. This is found in (5). Going back to sampling, if the mean of the parent population is the true value then the mean deviation of B times the sample means is less than the mean deviation of the mean of the sample means.

To be able to find the inequalities for B when other laws of error are used it is necessary to find the law for the mean when the law for the individual error law is known.

The limits for b are not always the best limits, for under Theorem 1 the lower limit for b is found by setting the first derivative of (2) equal to zero and solving for b. This gives the middle of the interval as the lower limit of b, which brings b nearer to unity.

Intervals for the constant b can be found wher. the higher moments are used.

In general the coefficients b and B are so near to unity that bm and BM do not differ appreciably from m and M respectively, yet this is not always the case, notably, when the standard error of m is large relative to m.

THE UNIVERSITY OF MICHIGAN

ON THE TRIGONOMETRIC EXPANSION OF ELLIPTIC FUNCTIONS*

BY M. A. BASOCO

1. *Introduction.* The problem of expressing an elliptic function in terms of infinite sums of trigonometric functions has been treated by Hermite,† Briot and Bouquet,‡ A. C. Dixon§ and others. In the present paper we treat the same problem from the point of view of Cauchy's residue theorem in function theory, which is also Briot and Bouquet's starting point, but we differ from these authors in that the integrand we use leads to an expansion for an elliptic function which is valid in an arbitrarily wide, but finite, strip of the complex plane, and which contains certain classical results as special cases. An interesting feature of our expansion is that it yields quite directly the Fourier series development of the function. Some examples of this property are indicated as an illustration of the applicability of our formula. It should be noted that the integrand used was first given by F. Gomes Teixeira¶ in another connection.

2. *Expansion of $f(z)$.* Let $f(z)$ be an elliptic function with periods π and $\pi\tau$, where $\tau=\mu+i\nu$, μ, ν real and $\nu>0$. Suppose that in a fundamental period cell $f(z)$ has k poles, $z=a_r$, $(r=1, 2, \cdots, k)$. Further, let the order of these poles be m_r, so that in the neighborhood of $z=a_r$, $f(z)$ has the Laurent expansion

$$(1)\quad f(z)=\frac{A_{m_r}^{(r)}}{(z-a_r)^{m_r}}+\cdots+\frac{A_2^{(r)}}{(z-a_r)^2}+\frac{A_1^{(r)}}{z-a_r}+P(z-a_r),$$

where $P(z-a_r)$ is a power series in $z-a_r$.

* Presented to the Society, December 30, 1930.

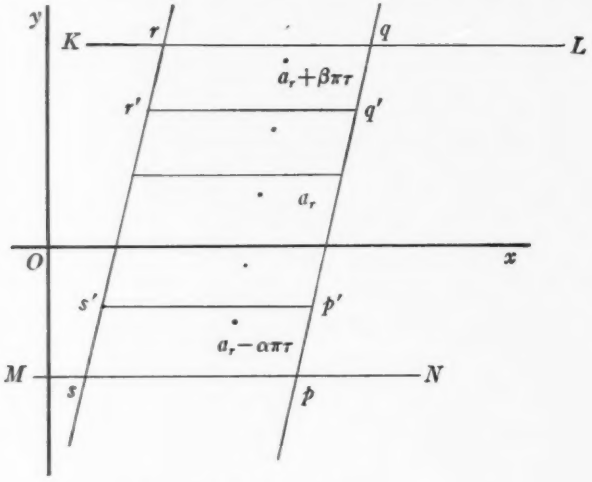
† C. Hermite, Annales de Toulouse, vol. 2 (1888). (See also Halphen, vol. 1, p. 461.)

‡ Briot and Bouquet, *Théorie des Fonctions Elliptiques*, 2d ed., 1875, p. 286.

§ A. C. Dixon, Quarterly Journal of Mathematics, vol. 34, p. 222.

¶ F. Gomes Teixeira, *Sur les séries ordonnées suivant les puissances d'une fonction donnée*, Journal für Mathematik, vol. 122, pp. 97–123.

Consider a parallelogram $pqrs$ consisting of $(\beta+1)$ period cells above the real axis and α below, so that we may write $|\overline{sp}| = \pi$ and $|\overline{pq}| = |\pi\tau|$ $(\alpha+\beta+1)$. In this parallelogram, $f(z)$ has $(\alpha+\beta+1)k$ poles which may be represented by the affixes



$$z = a_r + m\pi\tau, \quad \begin{cases} r = 1, 2, 3, \cdots, k; \\ m = -\alpha, -\alpha+1, \cdots, -1, 0, 1, 2, \cdots, \beta. \end{cases}$$

The function of t

$$(2) \qquad \phi(t) \equiv \frac{f(t)e^{2it}}{e^{2it}-e^{2iz}} = f(t)\left\{\frac{1}{2}+\frac{1}{2i}\operatorname{ctn}(t-z)\right\}$$

has, in the parallelogram $pqrs$, poles at $t=z$ and $t=a_r+m\pi\tau$; the corresponding residues are easily calculated and found to be

$$(3) \quad \frac{1}{2i}f(z), \text{ and } \frac{1}{2}A_1^{(r)} + \frac{1}{2i}\sum_{s=1}^{m_r}\frac{A_s^{(r)}}{(s-1)!}\mathcal{D}_{\omega_{r,m}}^{(s-1)}\operatorname{ctn}(\omega_{r,m}-z),$$

where $\mathcal{D}^{(s-1)}$ is the differential operator of order $(s-1)$ and $\omega_{r,m}$ is the argument $a_r+m\pi\tau$ which is to be substituted after the differentiation has been performed.

On integrating $\phi(t)$ around the contour C of the parallelogram $pqrs$ we obtain, by Cauchy's theorem, the expression

$$f(z) = \frac{1}{\pi}\int_{(C)} \frac{f(t)e^{2it}}{e^{2it} - e^{2iz}}\, dt + \sum_{s=1}^{m_r}\sum_{r=1}^{k}\sum_{m=-\alpha}^{\beta} \frac{A_s^{(r)}}{(s-1)!}\, \mathfrak{D}_{\omega_{r,m}}^{(s-1)} \operatorname{ctn}(z - \omega_{r,m}). \tag{4}$$

In writing (4) we have also used the fact that for an elliptic function the sum of its residues in any period cell vanishes.

The integral in (4) may be transformed into an infinite series in the following manner. Due to the fact that the integrand has the period π, the integrals along $\overline{pq}$ and $\overline{rs}$ cancel. Further, along $\overline{sp}$ we have

$$\frac{1}{e^{2it} - e^{2iz}} = \frac{1}{e^{2it}} + \frac{e^{2iz}}{e^{4it}} + \frac{e^{4iz}}{e^{6it}} + \cdots,$$

while along $\overline{rq}$,

$$\frac{1}{e^{2it} - e^{2iz}} = -\left(\frac{1}{e^{2iz}} + \frac{e^{2it}}{e^{4iz}} + \frac{e^{4it}}{e^{6it}} + \cdots\right).$$

It follows that our integral may be written in the form

$$\frac{1}{\pi}\left\{\int_{(\overline{sp})} f(t)\sum_{n=0}^{\infty} e^{2ni(z-t)}dt + \int_{(\overline{rq})} f(t)\sum_{n=1}^{\infty} e^{-2ni(z-t)}dt\right\}.$$

Interchanging the order of integration and summation, which is permissible, and using the notation

$$\begin{cases} C_n = \dfrac{1}{\pi}\displaystyle\int_{(\overline{sp})} f(t)e^{-2nit}dt, & (n = 0, 1, 2, \cdots); \\ C_{-n} = \dfrac{1}{\pi}\displaystyle\int_{(\overline{rq})} f(t)e^{2nit}dt, & (n = 1, 2, 3, \cdots), \end{cases} \tag{5}$$

we may write the formula (4) in the form

$$f(z) = \sum_{n=-\infty}^{n=\infty} C_n e^{2niz} + \sum_{s=1}^{m_r}\sum_{r=1}^{k}\sum_{m=-\alpha}^{m=\beta} \frac{A_s^{(r)}}{(s-1)!}\, \mathfrak{D}_{\omega_{r,m}}^{(s-1)} \operatorname{ctn}(z - \omega_{r,m}). \tag{6}$$

3. *Advantage of the Function $\phi(t)$.* The particular advantage in using the integrand $\phi(t)$ is that it leads in a natural manner to the expressions in (5) for the constants C_n and C_{-n}, all of which, with the exception of C_0, may be readily calculated. Thus, to compute C_n, integrate $f(t)e^{-2\pi it}$ around the contour $spp's'$ and apply Cauchy's theorem. Proceeding in the usual manner, it is readily found that the sum of the residues of the integrand, relative to the poles $t=a_r-\alpha\pi\tau$, has the value

$$q^{2n\alpha}\sum_{r=1}^{k}\sum_{s=1}^{m_r}\frac{(-2ni)^{s-1}A_s^{(r)}}{(s-1)!}e^{-2\pi ia_r}.$$

Furthermore, the integrals along pp' and ss' cancel while

$$\int_{(\overline{p's'})}f(t)e^{-2\pi it}\,dt=-q^{-2r}\int_{(\overline{sp})}f(t)e^{-2\pi it}dt,$$

where, as usual, $q=e^{\pi i\tau}$.

We therefore find that

$$(7)\qquad C_n=\frac{q^{2n(\alpha+1)}}{1-q^{2n}}\sum_{r=1}^{k}\sum_{s=1}^{m_r}\frac{(-2i)^s n^{s-1}A_s^{(r)}}{(s-1)!}e^{-2\pi ia_r},\qquad (n=1,2,3,\cdots).$$

It should be noticed that C_0 is left undetermined.

In a similar manner, applying the residue theorem to the function $f(t)e^{2\pi it}$, using the parallelogram $r'q'qr$ as contour of integration, we find that

$$(8)\qquad C_{-n}=\frac{q^{2n(\beta+1)}}{1-q^{2n}}\sum_{r=1}^{k}\sum_{s=1}^{m_r}\frac{(2i)^s n^{s-1}A_s^{(r)}}{(s-1)!}e^{2\pi ia_r}.$$

If now, (7) and (8) are substituted in (6) we obtain the result

$$(9)\qquad \begin{cases} f(z)=C_0+\displaystyle\sum_{r=1}^{k}\sum_{s=1}^{m_r}\sum_{n=1}^{\infty}\frac{(2i)^s n^{s-1}q^{2n}A_s^{(r)}}{(s-1)!(1-q^{2n})}\{q^{2n\beta}e^{-2\pi i(z-a_r)} \\ \qquad\qquad +(-1)^s q^{2n\alpha}e^{2\pi i(z-a_r)}\} \\ \qquad\qquad +\displaystyle\sum_{r=1}^{k}\sum_{s=1}^{m_r}\sum_{n=-\alpha}^{n=\beta}\frac{A_s^{(r)}}{(s-1)!}D_{\omega_{r,n}}^{(s-1)}\operatorname{ctn}(z-\omega_{r,n}), \end{cases}$$

which is valid in the strip bounded by the lines KL and MN. The width of this strip is, of course, determined by the values of the integers α and β.

4. *Some Special Cases.* The following special cases of (9) are of interest. First, suppose that all the poles of $f(z)$ are simple, so that $m_r=1$; further let $\alpha=0, \beta=0$. Then

$$f(z) = C_0 + 4\sum_{n=1}^{\infty}\sum_{r=1}^{k}\frac{q^{2n}A_1^{(r)}}{1-q^{2n}}\sin 2n(z-a_r) + \sum_{r=1}^{k}A_1^{(r)}\operatorname{ctn}(z-a_r), \tag{10}$$

which is another form of the classical formula of decomposition

$$f(x) = C_0 + \sum_{r=1}^{k}A_1^{(r)}\frac{\vartheta_1'(z-a_r)}{\vartheta_1(z-a_r)}, \tag{11}$$

where $\vartheta_1(z)$ is one of the Jacobi theta functions.

Again, let α and β become infinite; then since the absolute value of q is less than unity, it is easily seen that (9) becomes

$$f(z) = C_0 + \sum_{r=1}^{k}\sum_{s=1}^{m_r}\sum_{n=-\infty}^{n=\infty}\frac{A_s^{(r)}}{(s-1)!}D_{\omega_{r,n}}^{(s-1)}\operatorname{ctn}(z-\omega_{r,n}), \tag{12}$$

where

$$C_0 = \frac{1}{\pi}\int_{z_0}^{z_0+\pi} f(t)dt,$$

which is given by Briot and Bouquet (loc. cit. p. 291).

Various other forms may be given to formula (10). Thus, since we have the identity

$$i\operatorname{ctn}(z-a_r) \equiv \frac{2}{1-e^{2ni(z-a_r)}} - 1 = 1 + 2\sum_{n=1}^{\infty}e^{2ni(z-a_r)},$$

(provided $\eta>\mu_r$, where $z=\xi+i\eta$ and $a_r=\lambda_r+i\mu_r$), and since the sum of the residues is zero, we may write

$$f(z) = C_0 + 2i\sum_{n=1}^{\infty}\sum_{r=1}^{k}\frac{A_1^{(r)}}{1-q^{2n}}(q^{2n}e^{-2ni(z-a_r)} - e^{2ni(z-a_r)}), \tag{13}$$

which holds in a strip of the period cell such that $I(z) > I(a_r)$.

In a similar manner, (10) may be written in the form

$$(14)\quad f(z) = C_0 + 2i\sum_{n=1}^{\infty}\sum_{r=1}^{k}\frac{A_1^{(r)}}{1-q^{2n}}(e^{-2ni(z-a_r)} - q^{2n}e^{2ni(z-a_r)}),$$

which is valid provided $I(z) < I(a_r)$.

5. *Conclusion.* In this concluding section we shall use our formula (9) to obtain the Fourier series expansion of certain elliptic functions. We shall first consider the Weierstrass $\mathfrak{P}$ function, which in the neighborhood of the origin has the expansion

$$\wp(z) = \frac{1}{z^2} + \frac{g_2}{20}z^2 + \cdots.$$

If in (9) we put

$$\alpha = 0, \beta = 0, m_r = 2, A_1 = 0, A_2 = 1, a_r = 0,$$

we obtain at once

$$(15)\quad \wp(z, \pi, \pi\tau) = C_0 + \frac{1}{\sin^2 z} - 8\sum_{n=1}^{\infty}\frac{nq^{2n}}{1-q^{2n}}\cos 2nz.$$

The determination of C_0 follows immediately from the fact that

$$\lim_{z\to 0}\left(\wp(z) - \frac{1}{z^2}\right) = 0.$$

Thus, we obtain

$$(16)\quad C_0 = \frac{1}{3}\left(-1 + 24\sum_{n=1}^{\infty}\frac{nq^{2n}}{1-q^{2n}}\right) = \frac{1}{3}\frac{\vartheta_1'''}{\vartheta_1'}.$$

Next, let us consider the square of the $\wp$ function:

$$\wp^2(z) = \frac{1}{z^4} + \frac{g_2}{10} + c_2z^2 + \cdots.$$

If, in (9), we put

$$A_1 = A_2 = A_3 = 0, A_4 = 1; \alpha = \beta = 0; m_r = 4, a_r = 0$$

it will follow, after a slight reduction, that

$$(17)\qquad \wp^2(z, \pi, \pi\tau) = C_0 + \frac{1}{\sin^4 z} - \frac{2}{3\sin^2 z} + \frac{16}{3}\sum_{n=1}^{\infty}\frac{n^3q^{2n}}{1-q^{2n}}\cos 2nz.$$

The constant C_0 is now determined by the condition

$$\lim_{z\to 0}\left(\wp^2(z) - \frac{1}{z^4}\right) = \frac{g_2}{10},$$

and the fact that

$$g_2 = \frac{4}{3} + 320\sum_{n=1}^{\infty}\frac{n^3q^{2n}}{1-q^{2n}},$$

(see Tannery-Molk, vol. 4, p. 107). It is found that

$$(18)\qquad C_0 = \frac{1}{9}\left(1 + 240\sum_{n=1}^{\infty}\frac{n^3q^{2n}}{1-q^{2n}}\right).$$

As a last example we shall consider the square of the Jacobi sine-amplitude function and its reciprocal. In terms of the theta functions, these are essentially equivalent to

$$\frac{\vartheta_2^2\vartheta_3^2\vartheta_1^2(z)}{\vartheta_0^2(z)} \quad \text{and} \quad \frac{\vartheta_2^2\vartheta_3^2\vartheta_0^2(z)}{\vartheta_1^2(z)},$$

respectively. To obtain an expression for the first of these we put, in (9),

$$\alpha = \beta = 0;\ A_1 = 0,\ A_2 = 1;\ a_r = \frac{\pi\tau}{2};\ m_r = 2.$$

The constant C_0 is here determined by the fact that the function vanishes at the origin. After a slight calculation we obtain the expansion

$$(19)\qquad \vartheta_2^2\vartheta_3^2\frac{\vartheta_1^2(z)}{\vartheta_0^2(z)} = 8\sum_{n=1}^{\infty}\frac{nq^n}{1-q^{2n}} + \frac{1}{\sin^2\left(z-\frac{\pi\tau}{2}\right)} - 8\sum_{n=1}^{\infty}\frac{nq^{2n}}{1-q^{2n}}\cos 2n\left(z-\frac{\pi\tau}{2}\right).$$

which holds in a strip of the period cell such that $I(z) > I(a_r)$.

In a similar manner, (10) may be written in the form

$$(14)\quad f(z) = C_0 + 2i\sum_{n=1}^{\infty}\sum_{r=1}^{k}\frac{A_1^{(r)}}{1-q^{2n}}\left(e^{-2ni(z-a_r)} - q^{2n}e^{2ni(z-a_r)}\right),$$

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respectively. To obtain an expression for the first of these we put, in (9),

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The constant C_0 is here determined by the fact that the function vanishes at the origin. After a slight calculation we obtain the expansion

$$(19)\qquad \vartheta_2^2\vartheta_3^2\frac{\vartheta_1^2(z)}{\vartheta_0^2(z)} = 8\sum_{n=1}^{\infty}\frac{nq^n}{1-q^{2n}} + \frac{1}{\sin^2\left(z-\frac{\pi\tau}{2}\right)} - 8\sum_{n=1}^{\infty}\frac{nq^{2n}}{1-q^{2n}}\cos\ 2n\left(z-\frac{\pi\tau}{2}\right).$$

If we replace z by $z+\pi\tau/2$, we obtain

$$(20)\quad \vartheta_2^2\vartheta_3^2\frac{\vartheta_0^2(z)}{\vartheta_1^2(z)} = 8\sum_{n=1}^{\infty}\frac{nq^n}{1-q^{2n}} + \frac{1}{\sin^2 z} - 8\sum_{n=1}^{\infty}\frac{nq^{2n}}{1-q^{2n}}\cos 2nz.$$

From (19) we may derive a Fourier series in which the argument is z. Thus, since

$$\frac{1}{\sin^2\left(z-\frac{\pi\tau}{2}\right)} = \frac{-4qe^{-2iz}}{(1-qe^{-2iz})^2} = -4\sum_{n=1}^{\infty}nq^ne^{-2niz},$$

provided $\eta<\pi\nu/2$, where $z=\xi+i\eta$ and $\tau=\mu+i\nu$, we find, after some reduction,

$$(21)\quad \vartheta_2^2\vartheta_3^2\frac{\vartheta_1^2(z)}{\vartheta_0^2(z)} = 8\sum_{n=1}^{\infty}\frac{nq^n}{1-q^{2n}}(1-\cos 2nz),$$

which is valid in a strip bounded by lines through $\pm\pi\tau/2$ and parallel to the real axis.

THE UNIVERSITY OF NEBRASKA

A CONDENSED TABLE OF LINEAR FORMS*

BY P. H. DAUS

1. *Introduction.* In constructing his factor stencils,† D. N. Lehmer found it necessary to reconstruct the tables of linear forms for quadratic residues, due to the fact that all tables existing at that time contained errors.‡ The method of construction depended upon the observation that if a and b are numbers such that both belong or both do not belong to the form, then $c \equiv ab$, using the modulus of the form, belongs to the form. In particular, all powers of a number of the form belong to the form. The purpose of this paper is to indicate how a minimum number of entries may be made, from which all others may be obtained, and hence make it possible readily to check or construct the form for a given D. This is done with the aid of a table of odd primitive roots of primes, such tables being readily available.

2. *The Number of Required Entries.* It is well known that the linear forms are of the following types:

(1) $D \equiv 1 \pmod 4$, $2Dn + r_1, \cdots, r_{t_1}$, $t_1 = \phi(D)/2$,

(2) $D \equiv 2 \pmod 4$, $4Dn + r_1, \cdots, r_{t_2}$, $t_2 = 2\phi(D)$,

(3) $D \equiv 3 \pmod 4$, $4Dn + r_1, \cdots, r_{t_3}$, $t_3 = \phi(D)$,

where each r is an odd integer less than the indicated modulus, such that D is a quadratic residue of $2Dn+r$, whenever $2Dn+r$ is a prime for type (1), and similar statements for types (2) and (3), and where ϕ is the totient function, $\phi(D) = \phi(-D)$.

Let $D = \pm p_1 p_2 \cdots$, the p's being distinct primes, and let l be the least common multiple of p_1-1, p_2-1, $\cdots$; then the

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† Published by the Carnegie Institution of Washington, 1929. See also *An announcement regarding factor stencils*, this Bulletin, vol. 35 (1929), p. 684.

‡ For tables of linear forms see M. Kraitchik, *Théorie des Nombres*, vol. 1, 1922, pp. 164–186, and *Recherches sur la Théorie des Nombres*, vol. 1, 1924, pp. 205–215. For a list of errors in these tables see D. H. Lehmer, this Bulletin, vol. 35 (1929), p. 865. These corrections have been republished by Kraitchik in his *Recherches sur la Théorie des Nombres*, vol. 2, 1929, pp. 180–182.

maximum exponent to which any integer belongs modulo D is l, and the existence of such integers follows from the existence of odd primitive roots of primes. For if ρ_i is an odd primitive root of p_i, then $\rho_i+2k_ip_i$ is also a primitive root of p_i, and since the set of linear diophantine equations

$$\rho_1 + 2k_1p_1 = \rho_2 + 2k_2p_2 = \cdots = \rho \tag{4}$$

always has a solution, it is possible to find a common primitive root of the p's, and ρ belongs to the exponent l modulo D. It is not necessary that an integer r be a common primitive root to belong to the exponent l, for evidently each p_i-1 contains the factor 2, and this factor need be introduced into the exponent only once.

If r_1 be any integer belonging to the exponent l, such that $(D/r_1)=1$, then the powers of r_1 are all incongruent with respect to the modulus of the form and account for l entries of the form. If $t_j/l=m$, $(j=1,2,3)$, then it is merely necessary to seek m entries $r_1, \cdots, r_i, \cdots, r_m$, such that $r_i \not\equiv r_1^k$, $(k=1, \cdots, l)$, and such that $(D/r_i)=1$. If D is positive and if $r_1^{l/2} \not\equiv -1$, using the modulus of the form, then the number of required entries may be reduced by half by using the $\pm$ sign with each entry.

3. *Linear Forms for a Prime.* Case 1. If $D=\pm p\equiv 1 \pmod 4$, and if ρ is any odd primitive root of p, then evidently $r\equiv\rho^2$ belongs to the form and r is the required entry, r^k, $(k=1, \cdots, (p-1)/2)$, giving all entries of the form. Case 2. If $D=\pm p\equiv 3 \pmod 4$, and if ρ is any odd primitive root of p, then D is a residue of either ρ or $\rho+2p$. Call the proper one r. Then r belongs to the exponent $p-1=t_3$ and hence the form is indicated by $4Dn+r^k$.

4. *Linear Forms for D Composite.* The determination of linear forms for a composite D is made to depend upon one or both of the following principles, illustrated by the case when D is the product of two primes. The method may be extended to the product of any number of primes, or the forms for the product of three primes may be made to depend upon those for two primes and so on. If $D=p_1p_2$, consider first the case when p_1-1 and p_2-1 have no common factor other than 2. Then by inspection of simple tables or the method indicated by (4), a common primitive root may be found. This root belongs to the

exponent l and from it proper entries may be found as in Case 2 above. Suppose secondly that p_1-1 is a factor of p_2-1, and that ρ is an odd primitive root of p_2. Consider the set of integers

$$\text{(A)} \qquad \rho, \rho + 2p_2, \rho + 4p_2, \cdots,$$

the number of entries being p_1, $2p_1$ or $2p_1$ according as $D \equiv 1$, 2 or 3 (mod 4). If $D \equiv 1 \pmod 4$, this set taken modulo $2p_1$ will be the odd numbers from 1 to $2p_1-1$ in some order, and hence will contain exactly the required number of r's, namely, $(p_1-1)/2$. They are incongruent to each other modulo $2D$ and also to r_1, $(k=1, 2, \cdots, l=p_2-1)$. For $r_1 \equiv r_2 = r_1+2hp_2 \pmod{p_2}$ is impossible. Similarly, if $D \equiv 2$ or 3 (mod 4), the set taken modulo $4p_1$ will be the odd integers 1 to $4p_1-1$ in some order, and will contain exactly the required number, p_1-1, of r's, all of which are incongruent to each other and to r_1^k modulo $4D$.

In general if p_1-1 and p_2-1 have the common factor $2d$, we may form the set (A) from the primitive roots of p_2, and determine the r's. If $D \equiv 1 \pmod 4$, there will be $(p_1-1)/2$ of them, which divide into d subsets of $(p_1-1)/(2d)$ each, numbers of each subset being congruent modulo D, but incongruent to numbers of other subsets. For now $r_1^{(p_2-1)} \equiv 1$, and $r_1^k \equiv r_2 = r_1+2hp_2$ has solutions; the number being $l/(p_2-1) = (p_1-1)/(2d)$, all of which belong to one subset. Similarly, if $D \equiv 2$ or 3 (mod 4), the (p_1-1) r's divide into $2d$ subsets or, if we make $\pm$ entries, into d subsets. The selection of one r from each subset, r_1 being a common primitive root or at least belonging to exponent l, gives the complete set of entries.

5. *Numerical Illustrations.* In this manner a condensed table of linear forms may be constructed. We conclude with a few numerical illustrations based upon multiples of 13, the smallest odd primitive root of which is 7. The notation $2Dn+r_1^k\,(1, r_2, \cdots, r_m)$ indicates that the linear form is

$$2Dn + r_1, \cdots, r_1^k, r_2r_1, \cdots, r_2r_1^k, \cdots, r_mr_1^k.$$

$D=13\equiv 1 \pmod 4$, $7^2 \equiv 23 \pmod{26}$, $26n+23^k$.

$D=-13\equiv 3 \pmod 4$, $(D/7)=1$, $52n+7^k$.

$D=26$, (A) 7, 33, 59, 85; 59 and 85 are r's, but since $\rho^6 \not\equiv -1 \pmod{104}$, only one is needed. $85 \equiv -19$, hence $104n \pm 19^k$.

$D=-26$, (A) as above, 7 and 85 are r's, $104n+7^k(1, 85)$.

$D=39$, two entries required, $(39/7)=1$, $156n\pm 7^k$.

$D=-39$, (A) 7, 33, 59; $59=r$, $78n+59^k$.

$D=65$, (A) 7, 33, 59, 85, 111; 7 and 33 are r's, $\rho^6\equiv -1 \pmod{130}$, so that $\pm$ signs are not available. $130n+7^k(1, 33)$.

$D=-65$, (A) 7, 33, 59, 85, 111, 137, 163, 189, 215, 241; the r's are readily found to be those $\equiv 11, 13, 17, 19 \pmod{20}$; the form is $260n+33^k(1, 59, 111, 137)$.

$D=143=11\times 13$, $l=60$, $m=2$, 7 is a common primitive root and so is 41, 7 is not an r, $7+2\times 143=293$ is, 41 is an r. Hence we may use either $572n\pm 293^k$ or $572n\pm 41^k$.

$D=-247=-13\times 19$, $l=36$, $m=3$, the r's of the set (A) are indicated by underscoring, while 33 is found to be a common primitive root.

(A) 7, $\underline{33}$, $\underline{59}$, 85, 111, 137, 163, $\underline{189}$, 215, $\underline{241}$, 267, $\underline{293}$, $\underline{319}$, $\underline{345}$, $\underline{371}$, 397, 423, $\underline{449}$, 475. The nine r's divide into three subsets by making use of 33^{12}, namely

$$(33, 59, 345), \quad (189, 293, 449), \quad (241, 319, 371),$$

and hence the form is

$$494n+33^k(1, 189, 241).$$

University of California at Los Angeles

OFFICIAL COMMUNICATIONS

Meetings of the Society have been fixed at the following times and places:

NEW YORK CITY, February 28, 1931.

Abstracts must be in the hands of the Secretary of the Society, Professor R. G. D. Richardson, 501 West 116th St., New York City, not later than February 7. In order to be printed in the Bulletin in advance of the meeting, astracts must be in the hands of the Secretary not later than January 6. By invitation of the program committee, Professor C. R. Adams will deliver an address entitled *Linear q-difference equations.*

NEW YORK CITY, April 3-4, 1931.

Abstracts must be in the hands of the Secretary of the Society, Professor R. G. D. Richardson, 501 West 116th St., New York City, not later than March 13. In order to be printed in the Bulletin in advance of the meeting, the abstracts must be in the hands of the Secretary not later than March 6. There will be a symposium on the theory of groups in which Professors H. F. Blichfeldt, L. P. Eisenhart, W. A. Manning, J. A. Schouten, and others will participate.

CHICAGO, April 3-4, 1931.

Abstracts must be in the hands of the Associate Secretary, Mark H. Ingraham, University of Wisconsin, Madison, Wisconsin, not later than March 12. In order to be printed in the Bulletin in advance of the meeting, the abstracts must be in the hands of the Associate Secretary not later than March 6. By invitation of the program committee, Professor G. Y. Rainich will deliver an address entitled *Some recent applications of mathematics to physics.*

THE UNIVERSITY OF CALIFORNIA, BERKELEY, CALIFORNIA, April 11, 1931.

Abstracts must be in the hands of the Associate Secretary, T. M. Putnam, University of California, Berkeley, California, not later than March 21. In order to be printed in the Bulletin in advance of the meeting, the abstracts must be in the hands of the Associate Secretary not later than March 6.

R. G. D. RICHARDSON, *Secretary of the Society.*

Articles for insertion in the BULLETIN should be addressed to E. R. HEDRICK, Editor of the BULLETIN, University of California at Los Angeles. REVIEWS should be sent to W. R. LONGLEY, Yale University, New Haven, Conn. Notes should be sent to H. W. KUHN, Ohio State University, Columbus, Ohio.

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Changes of address of members should be communicated at once to the Secretary of the Society, R. G. D. Richardson, 501 West 116th St., New York.

The initiation fees and the annual dues of members of the Society (see this BULLETIN, p. 322, May 1930; and the List of Officers and Members, October, 1930, p. 58), are payable to the Treasurer of the Society, G. W. Mullins, 501 West 116th St., New York.

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